Superfluids Under Rotation
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TWISTED VORTEX STATE

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Content

1) Does the twisted state exist at all?

Hydrodynamic theory
- uniform twist
- linear theory of nonuniform twist

2) Generation of twisted vortex states

Numerical simulations

3) Observation in superfluid ^{3}He

sketch of twisted vortex lines

Twisted vortex states in classical fluids

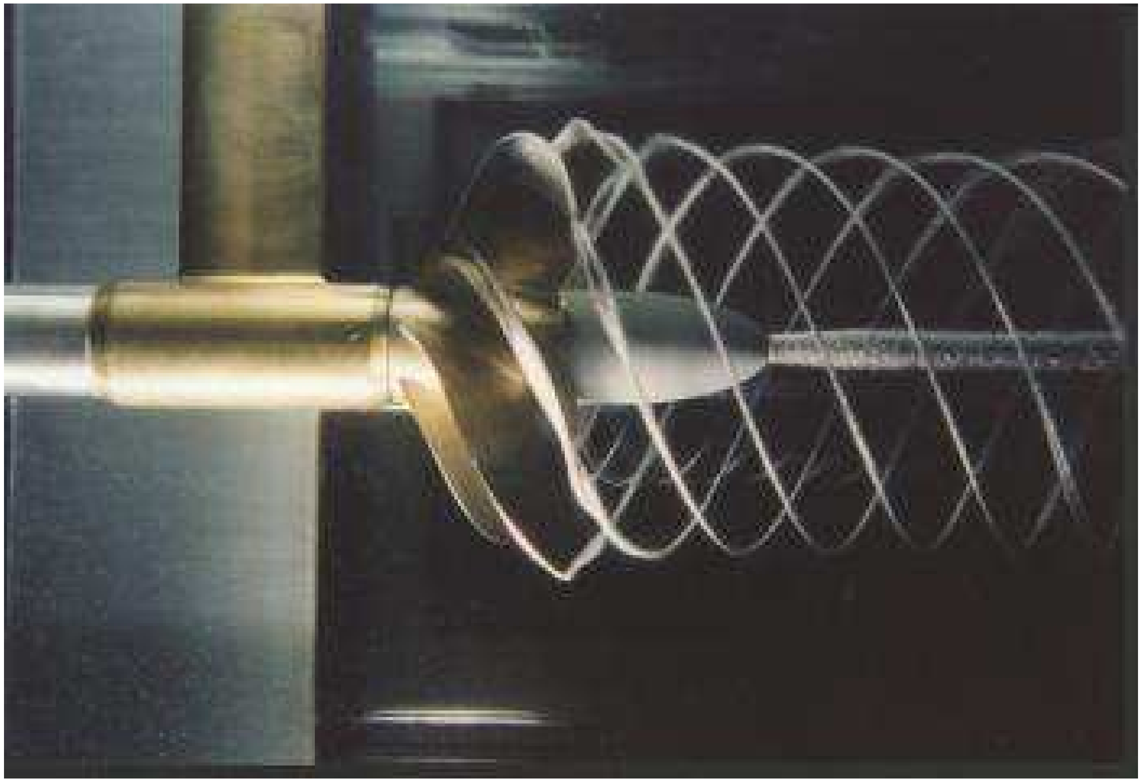
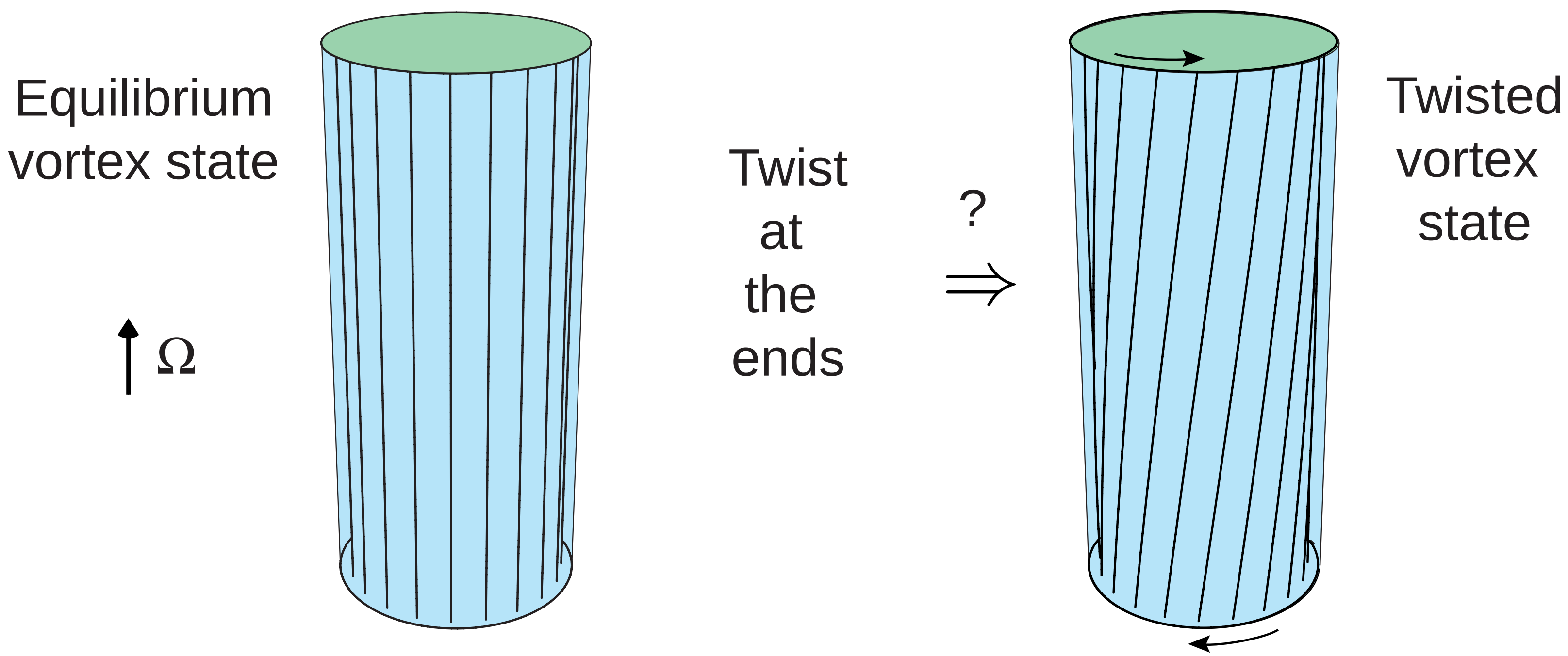


Figure: http://www.amc.edu.au/research/areas/cavitation/projects/

Stability of a polygon of helical vortices (Okulov 2004)

Rotating superfluid



Previous literature: Hall (1958), Andronikashvili et al (1961), Glaberson et al (1974), Sonin (1987), Donnelly (1991)...

→ no mention of twisted vortices!

Taylor-Proudman theorem: "Any slow motion in rotating fluid is columnar"

1) Does the twisted state exist?

Hydrodynamic equations

Superfluid velocity $\boldsymbol{v}_\mathsf{s}$

$$\nabla \times \boldsymbol{v}_\mathsf{s} = 0 \qquad \text{except at vortex lines}$$
$$\nabla \cdot \boldsymbol{v}_\mathsf{s} = 0$$

$\Rightarrow$ Vortex lines fully determine $\boldsymbol{v}_s(\boldsymbol{r}, t)$.

Line velocity $\boldsymbol{v}_\mathsf{L}$

$$\boldsymbol{v}_\mathsf{L} = \boldsymbol{v}_\mathsf{s}$$

Add mutual friction

$$\boldsymbol{v}_\mathsf{L} = \boldsymbol{v}_\mathsf{s} + \alpha \hat{\boldsymbol{l}} \times (\boldsymbol{v}_\mathsf{n} - \boldsymbol{v}_\mathsf{s}) - \alpha' \hat{\boldsymbol{l}} \times [\hat{\boldsymbol{l}} \times (\boldsymbol{v}_\mathsf{n} - \boldsymbol{v}_\mathsf{s})] \tag{1}$$

Continuum model of vorticity

(Hall and Vinen 1956)

$\boldsymbol{v}_{\mathsf{s}} = \langle \boldsymbol{v}_{\mathsf{s}}^{\mathsf{local}} \rangle$

$$\boldsymbol{\nabla} \times \boldsymbol{v}_{\mathsf{s}} = \boldsymbol{\omega}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_{\mathsf{s}} = 0$$

Line velocity $\boldsymbol{v}_{\mathsf{L}}$

$$\boldsymbol{v}_{\mathsf{L}} = \tilde{\boldsymbol{v}}_{\mathsf{s}} + \alpha \hat{\boldsymbol{l}} \times (\tilde{\boldsymbol{v}}_{\mathsf{n}} - \boldsymbol{v}_{\mathsf{s}}) - \alpha' \hat{\boldsymbol{l}} \times [\hat{\boldsymbol{l}} \times (\boldsymbol{v}_{\mathsf{n}} - \tilde{\boldsymbol{v}}_{\mathsf{s}})] \qquad (2)$$

where $\tilde{\boldsymbol{v}}_{\mathsf{s}} = \boldsymbol{v}_{\mathsf{s}} + \nu \boldsymbol{\nabla} \times \hat{\boldsymbol{\omega}}$, $\nu = (\kappa/4\pi)\,\ln(b/a)$.

Alternatively, one can use equation of motion for $\boldsymbol{v}_{\mathsf{s}}$:

$$\frac{\partial \boldsymbol{v}_s}{\partial t} = \boldsymbol{v}_s \times \boldsymbol{\omega} + \nu(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{\omega}} + \boldsymbol{\nabla}\phi$$

Uniformly twisted vortex state

Most symmetric state [cylindrical coordinates (r, ϕ, z)]

$$\boldsymbol{v}_s = v_\phi(r)\hat{\boldsymbol{\phi}} + v_z(r)\hat{\boldsymbol{z}},$$

$\Rightarrow$ vorticity

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{v}_s = \frac{1}{2}\left[-v_z'\hat{\boldsymbol{\phi}} + \left(\frac{v_\phi}{r} + v_\phi'\right)\hat{\boldsymbol{z}}\right]$$

Calculate vortex line velocity from (2). For a stationary state the radial velocity must vanish. This implies

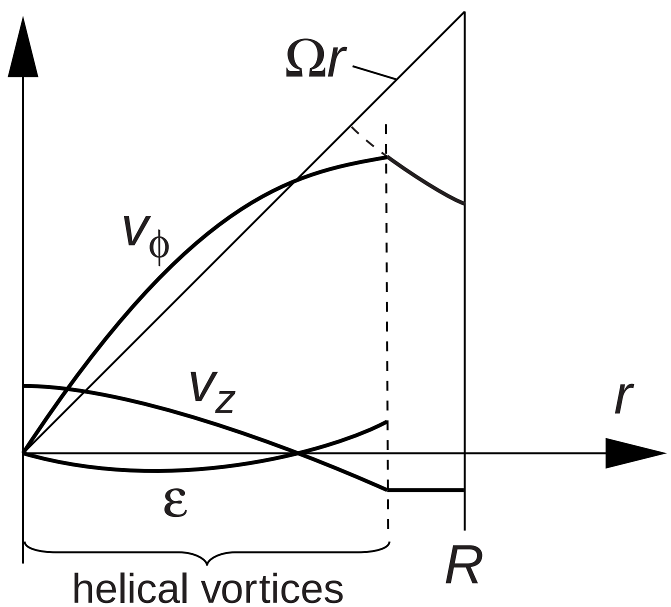
$$(\Omega r - v_\phi)\left(\frac{v_\phi}{r} + \frac{dv_\phi}{dr}\right) - v_z\frac{dv_z}{dr} + \frac{\nu}{|\omega| r}\left(\frac{dv_z}{dr}\right)^2 = 0.$$

This implies that the helical vortices rotate together with the normal fluid, $\boldsymbol{v}_L = \boldsymbol{v}_n = \boldsymbol{\Omega} \times \boldsymbol{r}$.

$\Rightarrow$ There exists a family of stationary, uniformly twisted states.

In a finite cylinder the total axial current must vanish,

$$\int_0^R dr\, r v_z = 0. \tag{3}$$



The functions $v_z(r)$, $v_\phi(r)$ and the radial displacement of the vortices compared to equilibrium state, $\epsilon(r)$, are sketched in the figure.

The simplest case is helical vortices with a wave vector $Q(r) = \text{constant}$. This has

$$\begin{aligned} v_\phi(r) &= \frac{(\Omega + Q v_0) r}{1 + Q^2 r^2}, \\ v_z(r) &= \frac{v_0 - Q\Omega r^2}{1 + Q^2 r^2}. \end{aligned} \tag{4}$$

Linearized hydrodynamics

Assume general velocity with circular symmetry

$$\boldsymbol{v}_s = v_r(r,z,t)\hat{\boldsymbol{r}} + v_\phi(r,z,t)\hat{\boldsymbol{\phi}} + v_z(r,z,t)\hat{\boldsymbol{z}}$$

Assume small deviation from rotating equilibrium.
$\Rightarrow$ waves of the form

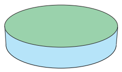
$$v_r = ckJ_1(\beta r)\,\exp(ikz - i\sigma t)$$
$$v_z = ic\beta J_0(\beta r)\,\exp(ikz - i\sigma t)$$

Dispersion relation [Glaberson, Johnson and Ostermeier (1974), Henderson and Barenghi (2004)]

$$\frac{\sigma}{\Omega} = \frac{-i\alpha(\beta^2 + 2k^2\eta_2) \pm i\sqrt{\alpha^2\beta^4 - 4(1-\alpha')^2k^2(\beta^2+k^2)\eta_1\eta_2}}{\beta^2 + k^2}$$

where $\eta_1 = 1 + \nu k^2/2\Omega$ and $\eta_2 = 1 + \nu(\beta^2 + k^2)/2\Omega$.

In order to understand the dispersion, we study special cases.

1) $\beta \to 0$, corresponds to a short cylinder 

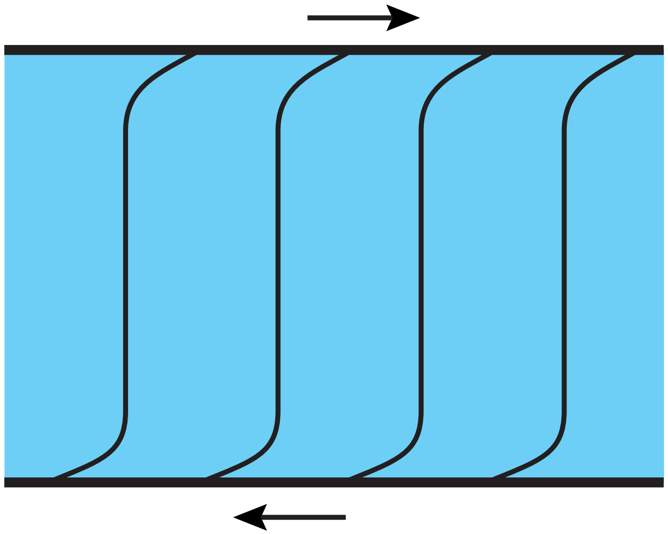
$\Rightarrow$ 2 Kelvin wave modes (Hall 1958)

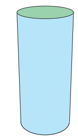
$$k_{\pm} = i\sqrt{\frac{2\Omega \pm \sigma}{\nu}}$$

and an inertial mode

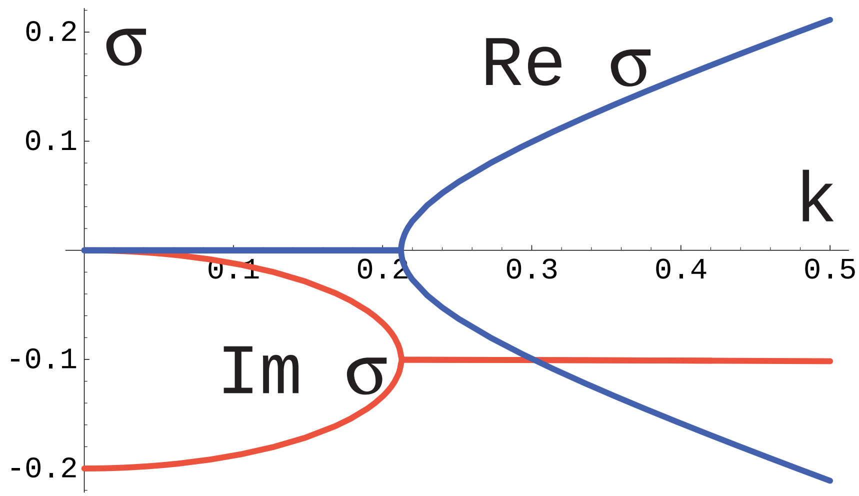
$$k_{\mathrm{i}} = 0$$

At low frequency ($\sigma \ll \Omega$) these give just the columnar motion because Kelvin waves are evanescent. No twisted state.



2) $k \to 0$, corresponds to a long cylinder 

$\Rightarrow$ 2 modes



The point $k = \sigma = 0$ corresponds to uniform twist!

At finite k the twist obeys diffusion equation

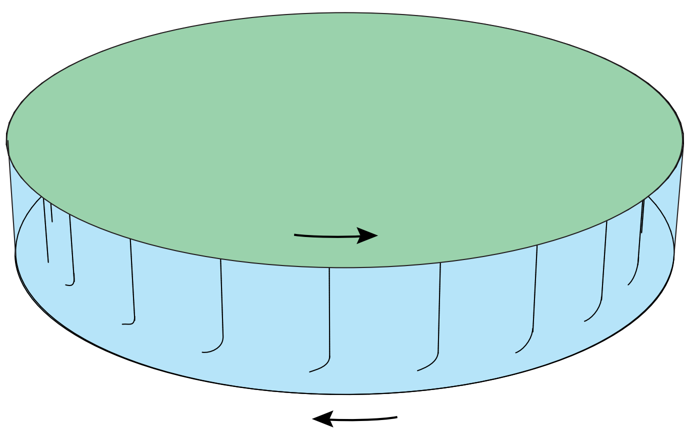
$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial z^2}, \quad D = \frac{1}{d} \left(\frac{2\Omega}{\beta^2} + \nu \right) \tag{5}$$

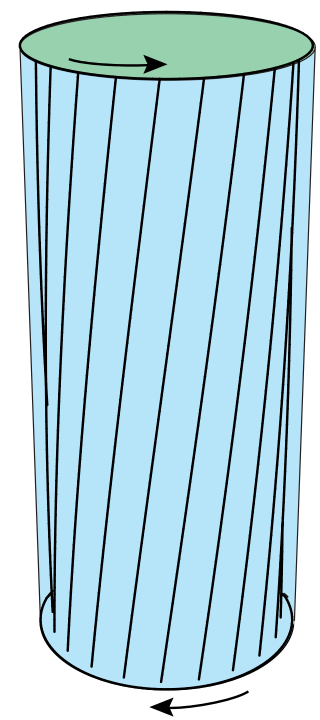
where $f(z, t) = v_r$ or v_z.

Summary of two opposite limits

Long cylinder
- twisted vortices

Parallel plates
- columnar vortices

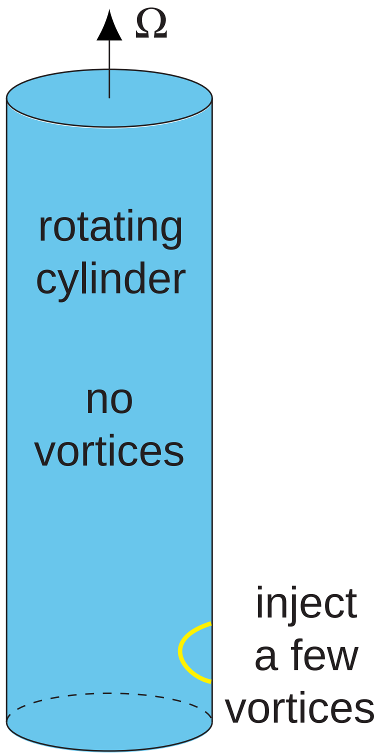


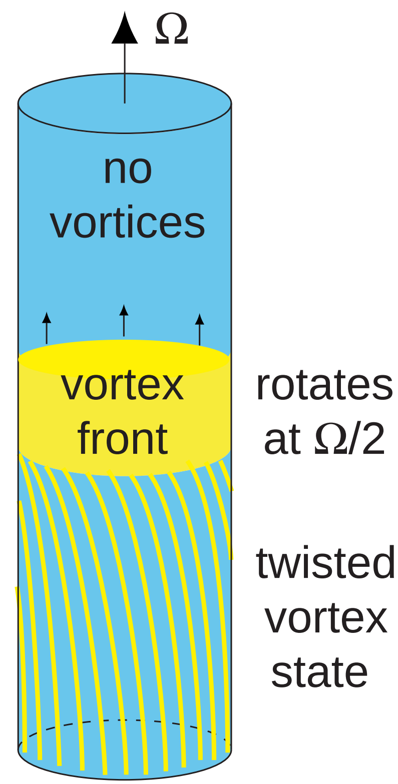


2) Generation of twisted vortex states

- superfluid in a cylinder
- cylinder rotating at $\Omega > \Omega_c$, but no vortices in the initial state
- generate vortices at one place

- vortices propagate along the cylinder and
- vortex ends rotate around the cylinder axis





Why vortex ends rotate?

Normal component rotates at $\boldsymbol{v}_n = \boldsymbol{\Omega} \times \boldsymbol{r}$.

Superfluid component: vortex lines move with the average superfluid velocity

1) vortex state: $\boldsymbol{v}_s \approx \boldsymbol{\Omega} \times \boldsymbol{r}$
$\Rightarrow$ vortex lattice rotates at angular velocity Ω

2) no vortices: $\boldsymbol{v}_s = 0$

3) vortex front
average superfluid angular velocity $\Omega/2 \Rightarrow$ vortex ends rotate at angular velocity $\Omega/2$

$\Rightarrow$ propagating vortex ends lag behind

Numerical simulation

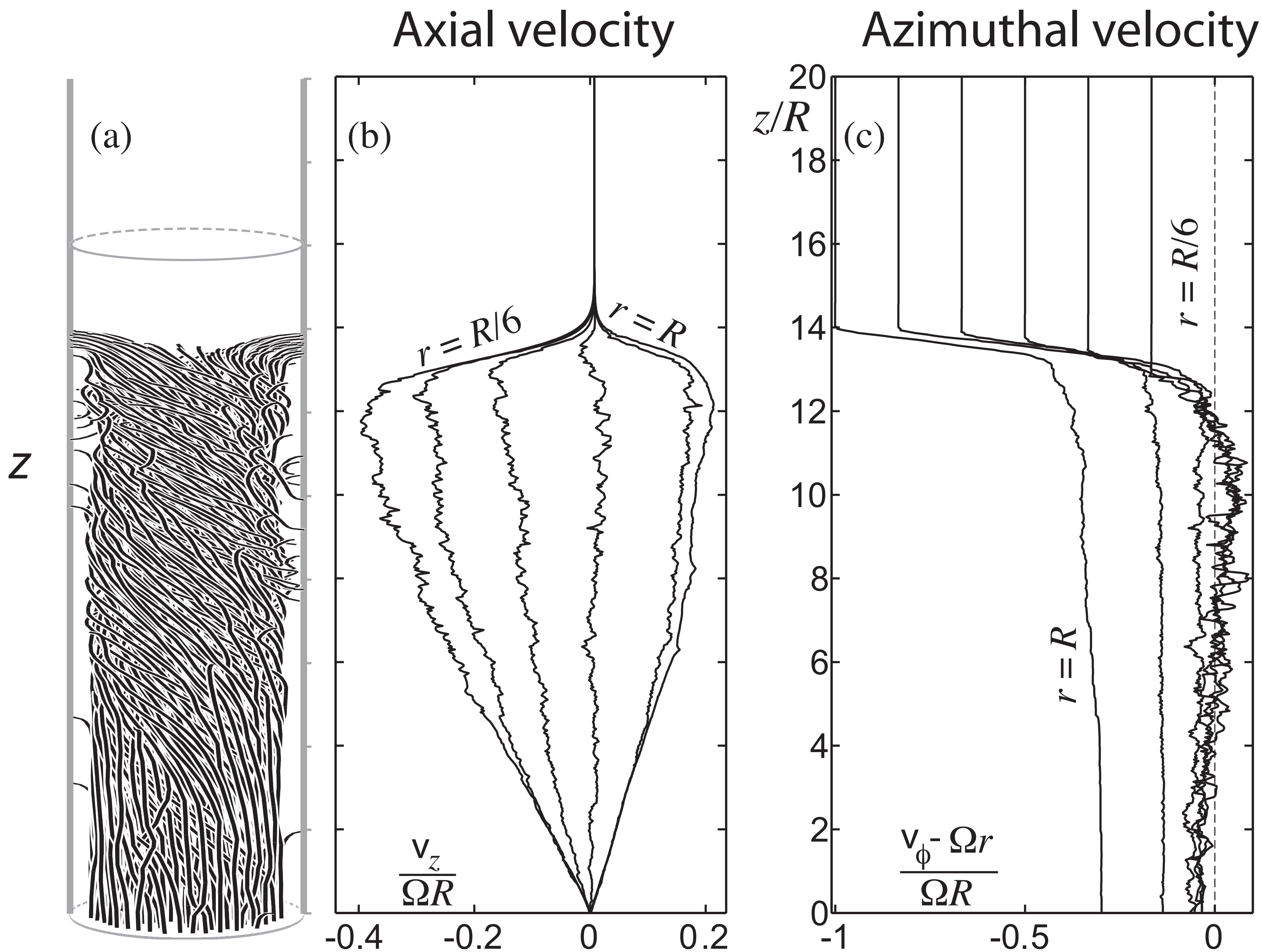
Vortex line velocity (2)

$$\boldsymbol{v}_L = \boldsymbol{v}_s + \alpha' \hat{\boldsymbol{l}} \times [(\boldsymbol{v}_n - \boldsymbol{v}_s) \times \hat{\boldsymbol{l}}] + \alpha \hat{\boldsymbol{l}} \times (\boldsymbol{v}_n - \boldsymbol{v}_s).$$

$\boldsymbol{v}_s$ is calculated from Biot-Savart integral. (Risto Hänninen)

The front and the twisted state is confirmed by numerical calculation

movie



Axial velocity
Azimuthal velocity
(a)
z
(b)
r = R/6
r = R
$\frac{v_z}{\Omega R}$
-0.4
-0.2
0
0.2
(c)
z/R
20
18
16
14
12
10
8
6
4
2
0
r = R/6
r = R
$\frac{v_\phi - \Omega r}{\Omega R}$
-1
-0.5
0

movie

Main observations

- the twisted state has axial current.
- individual vortices become unstable to generate Kelvin waves at large axial current
- the vortices glide at the bottom plate

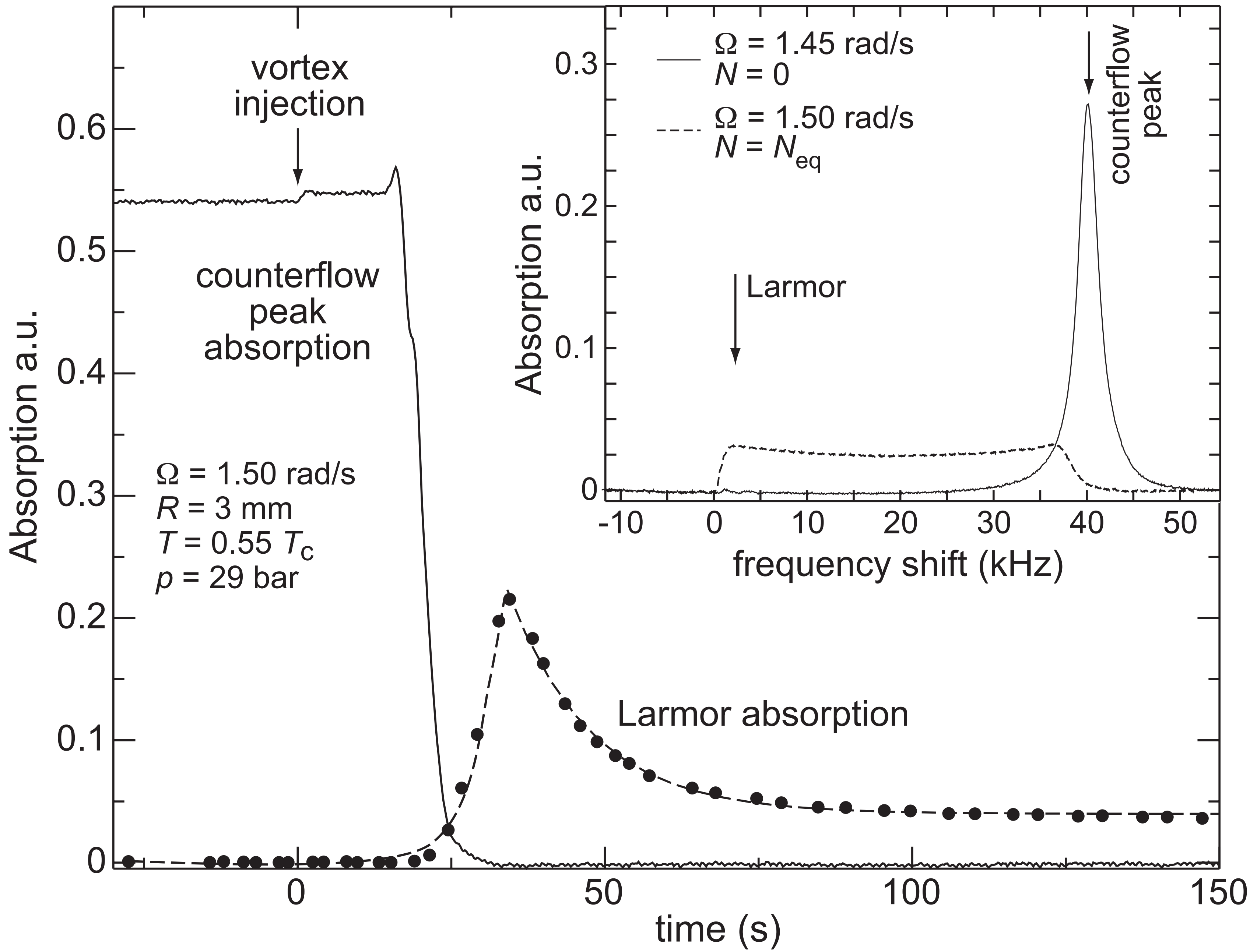
$\Rightarrow$ relaxation of the twist

- the relaxation is determined by the diffusion equation.

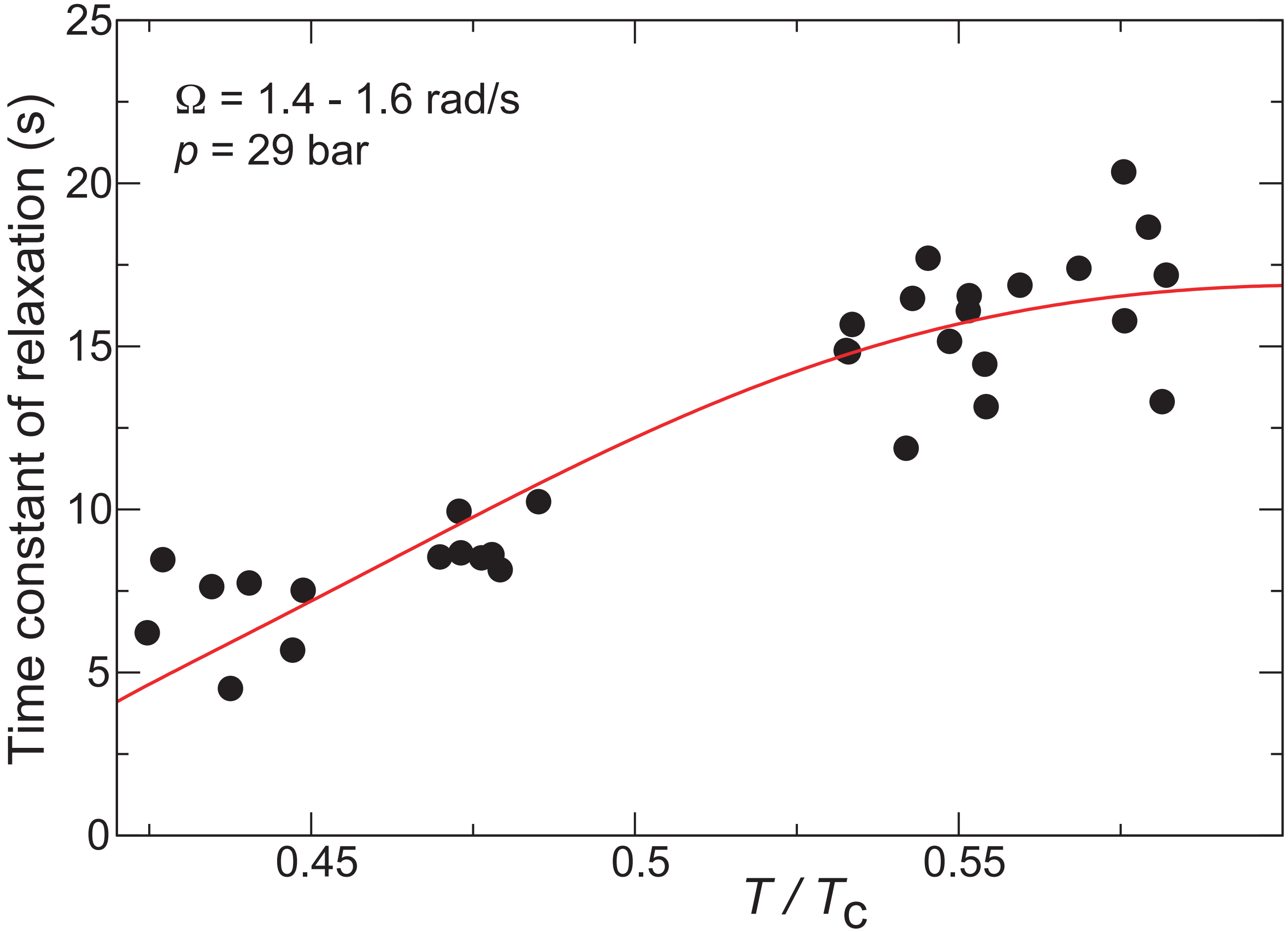
3) Experiment in superfluid ^{3}He-B

Vortex state was generated as discussed above.

The axial velocity v_z affects the texture, which is seen by NMR.



vortex
injection
counterflow
peak
absorption
Ω = 1.50 rad/s
R = 3 mm
T = 0.55 Tc
p = 29 bar
Larmor absorption
Absorption a.u.
time (s)
Ω = 1.45 rad/s
N = 0
Ω = 1.50 rad/s
N = Neq
counterflow
peak
Larmor
frequency shift (kHz)



Diffusion constant

$$D = \frac{1}{d}\left(\frac{2\Omega}{\beta^2} + \nu\right) \propto \frac{1}{\text{mutual friction constant}} \tag{6}$$

Conclusions

Twisted vortex state is a possible state in long rotating cylinders.

The twisted state can be generated by vortex injection.

The twisted state has been seen in superfluid ^{3}He-B.

Eltsov et al, Phys. Rev. Lett. 96, 215302 (2006)