Simple understanding of the effective mass

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This note gives a simple classical interpretation of the semiclassical equations of electron motion. After writing this, I found out that Kittel, in Introduction to solid state physics, has very similar discussion of the effective mass.

Let us consider an object, we call it a particle, in a medium. (At this stage it is easiest to consider the case that the medium particles are different from the object considered, but we wish to consider also the case where the medium consists of the same type of particles as the object is.) The particle has bare mass m_b and velocity $\boldsymbol{v}$. It drags with itself the medium. As a result, the total momentum $\boldsymbol{p}$ differs from $m_b\boldsymbol{v}$. For the following, it is convenient to consider the velocity as a function of momentum, $\boldsymbol{v}(\boldsymbol{p})$. For simplicity, we limit to consider the isotropic case, where $\boldsymbol{v}$ and $\boldsymbol{p}$ are parallel. In order to be quantitative, we define *effective mass* by the relation

$$\frac{1}{m^*} = \frac{dv}{dp}. \tag{1}$$

Using this we can write expressions for the bare momentum $\boldsymbol{p}_b = m_b\boldsymbol{v}$ of the particle, and the momentum carried by the medium $\boldsymbol{p}_m = \boldsymbol{p} - m_b\boldsymbol{v}$,

$$\boldsymbol{p}_b = \int_0^p \frac{m_b}{m^*} dp \hat{\boldsymbol{p}}, \qquad \boldsymbol{p}_m = \left(p - \int_0^p \frac{m_b}{m^*} dp \right) \hat{\boldsymbol{p}}, \tag{2}$$

where $\hat{\boldsymbol{p}}$ is a unit vector in the direction of $\boldsymbol{p}$. In the special case that m^* is constant this reduces to

$$\boldsymbol{p}_b = \frac{m_b}{m^*}\boldsymbol{p}, \qquad \boldsymbol{p}_m = \left(1 - \frac{m_b}{m^*}\right)\boldsymbol{p} \tag{3}$$

Three cases can be distinguished. 1) The total momentum and velocity are in the same direction but medium momentum is in the opposite direction. This can be called *backflow* of the medium. In the case of constant m^* this corresponds to $0 < m^* < m_b$. 2) All three momenta are in the same direction. This can be called *drag* of the medium. In the case of constant m^* this corresponds to $m^* > m_b$. 3) The total momentum is opposite to the velocity. In the case of constant m^* this corresponds to $m^* < 0$. The last alternative apparently can take place only if quantum mechanics plays an essential role. In the quantum case $\boldsymbol{v}$ denotes an expectation value of velocity. Otherwise, all equations are valid in both the quantum and the classical case.

Let an external force $\boldsymbol{F}$ be applied to the particle. In addition to the particle, it also accelerates the medium. From Newton's law

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}. \tag{4}$$

Let $E(p)$ be the energy of the state of the excitation. We get

$$dE = \boldsymbol{F} \cdot d\boldsymbol{x} = \frac{d\boldsymbol{p}}{dt} \cdot d\boldsymbol{x} = \boldsymbol{v} \cdot d\boldsymbol{p}, \tag{5}$$

which gives

$$\boldsymbol{v} = \frac{dE}{d\boldsymbol{p}}. \tag{6}$$

Notice that equations (4) and (6) are the *semiclassical equations of electron motion*[1]. Thus these equations can be interpreted simply as *classical equations taking into account that part of the momentum is carried by the medium.*

From above we can solve relations between m^* and E:

$$\frac{1}{m^*} = \frac{d^2E}{dp^2}, \qquad E(p) = \int_0^p dp' \int_0^{p'} dp'' \frac{1}{m^*(p'')}. \tag{7}$$

Based on the above, let us consider an electron on a lattice (neglect other conduction electrons). The momentum that does not go to the electron has to go to the lattice. If the lattice is infinitely rigid, this means that the whole lattice has to move, although with a rather small velocity since the lattice is much heavier than the electron. In reality, the lattice is not infinitely rigid, which means that a deformation of the lattice has to follow the electron. If a constant force is applied to the electron, the electron makes Bloch oscillations. This means that on the average all the momentum has to go to the lattice. Thus the lattice accelerates at a constant (but small) rate. In practice, however, the force can be applied electrostatically. Assuming the whole system neutral, there is positive charge in the lattice, and the force on that just cancels the acceleration of the lattice on the average.

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[1] N. Ashcroft ja D. Mermin, Solid state physics
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