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Vortex Structures in Superfluids: Examples in ^{3}He

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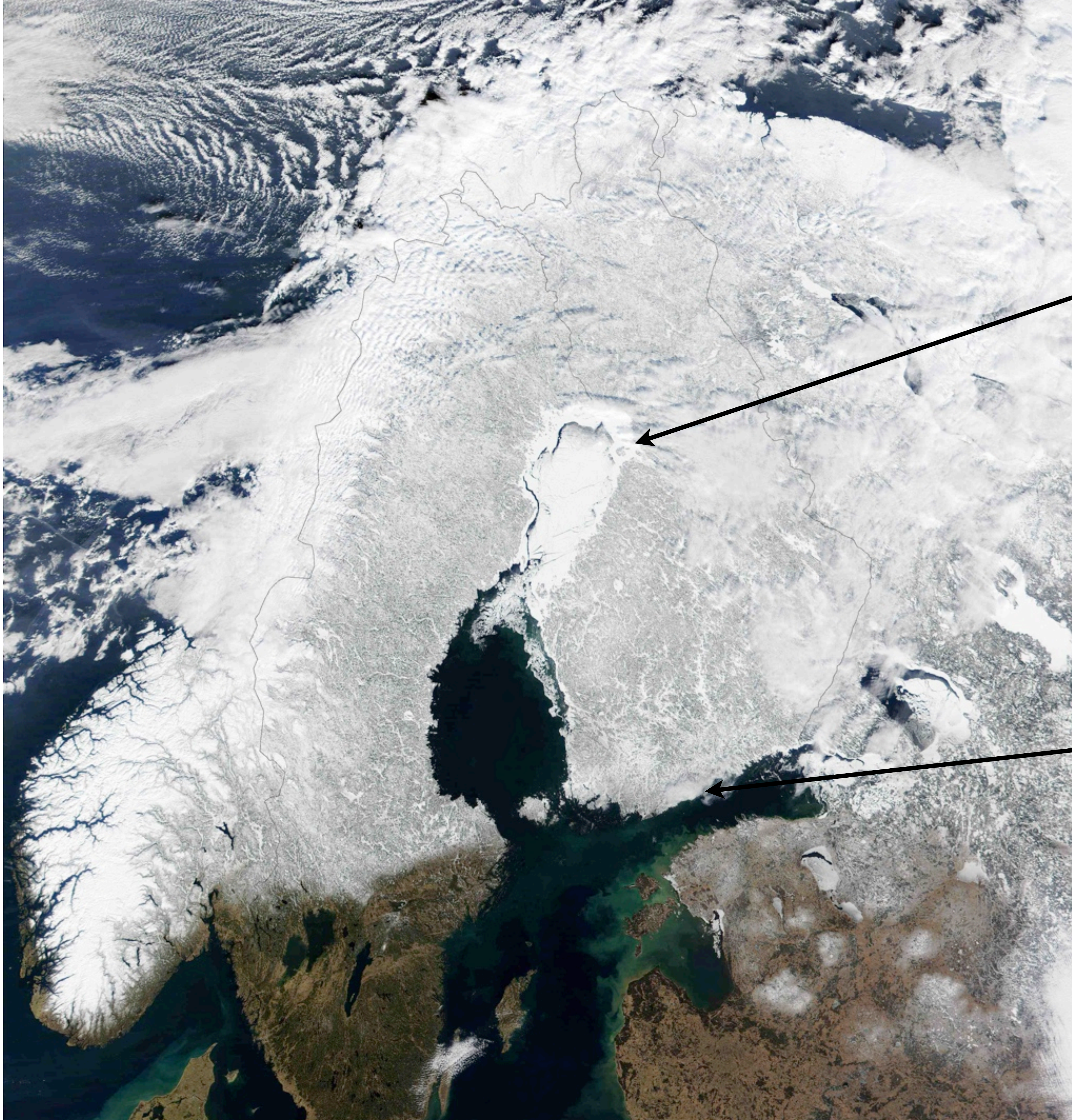

picture: Nasa

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Introduction to vortices

Multicomponent order parameter

Symmetry classification of vortices

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Summary of vortex structures in ^{3}He

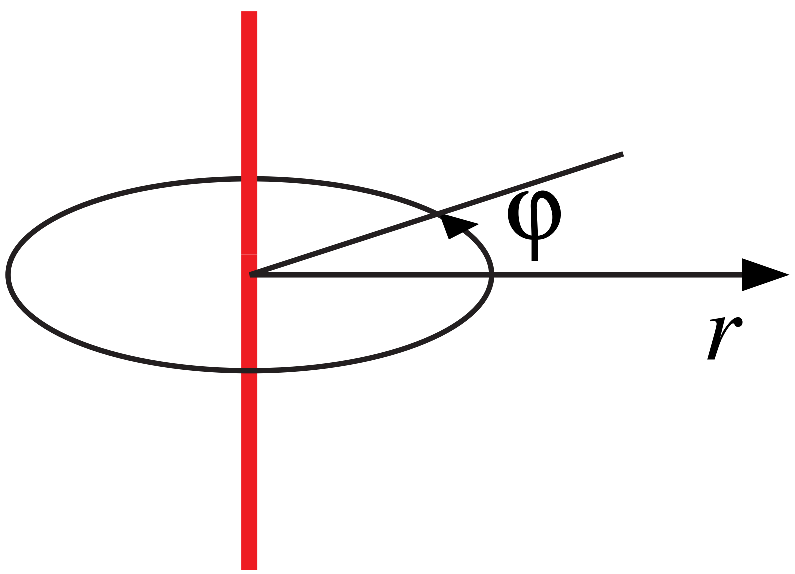
Simple superfluids

Order parameter is a scalar
- ^{4}He
- many superconductors Al, Nb, Pb, ...
- many condensates of dilute gases

$$\Psi(\mathbf{r}) = \Psi_0 \exp[i\phi(\mathbf{r})] \qquad \boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\phi$$

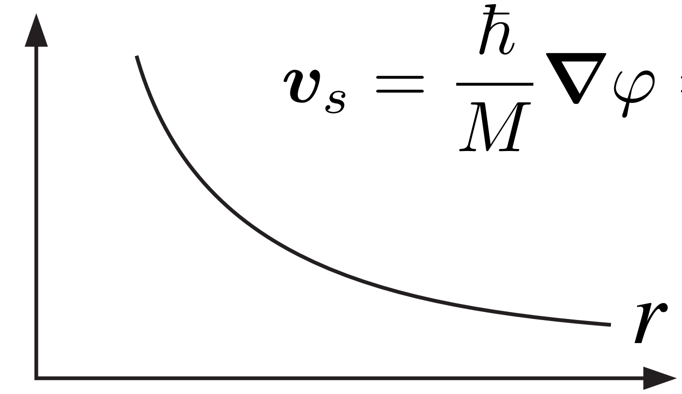
Vortex line

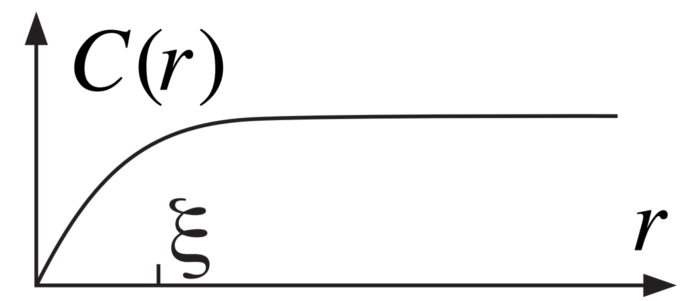
$$\Psi(r,\varphi,z) = \Psi_0 e^{\mathrm{i}\varphi} C(r)$$

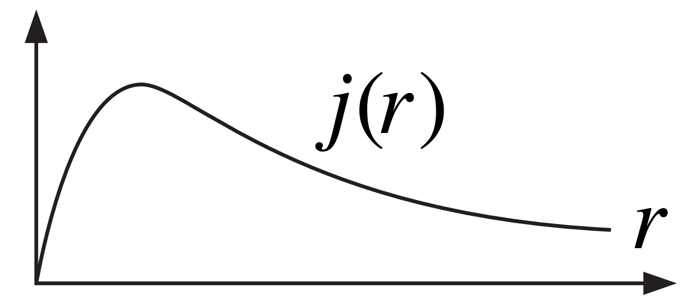


$$\boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\varphi = \frac{\hbar}{M}\,\frac{\hat{\boldsymbol{\varphi}}}{r}$$

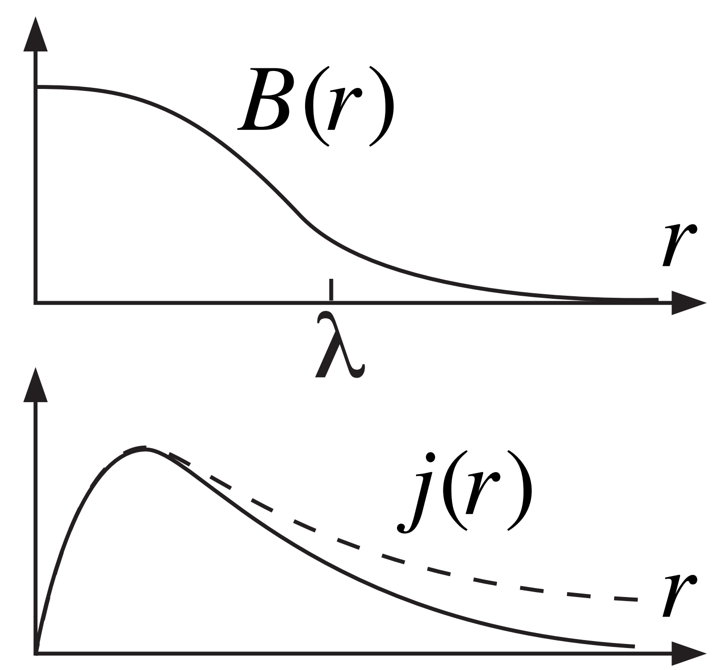
$$\kappa = \oint d\boldsymbol{l}\cdot\boldsymbol{v}_s = \frac{h}{M}$$



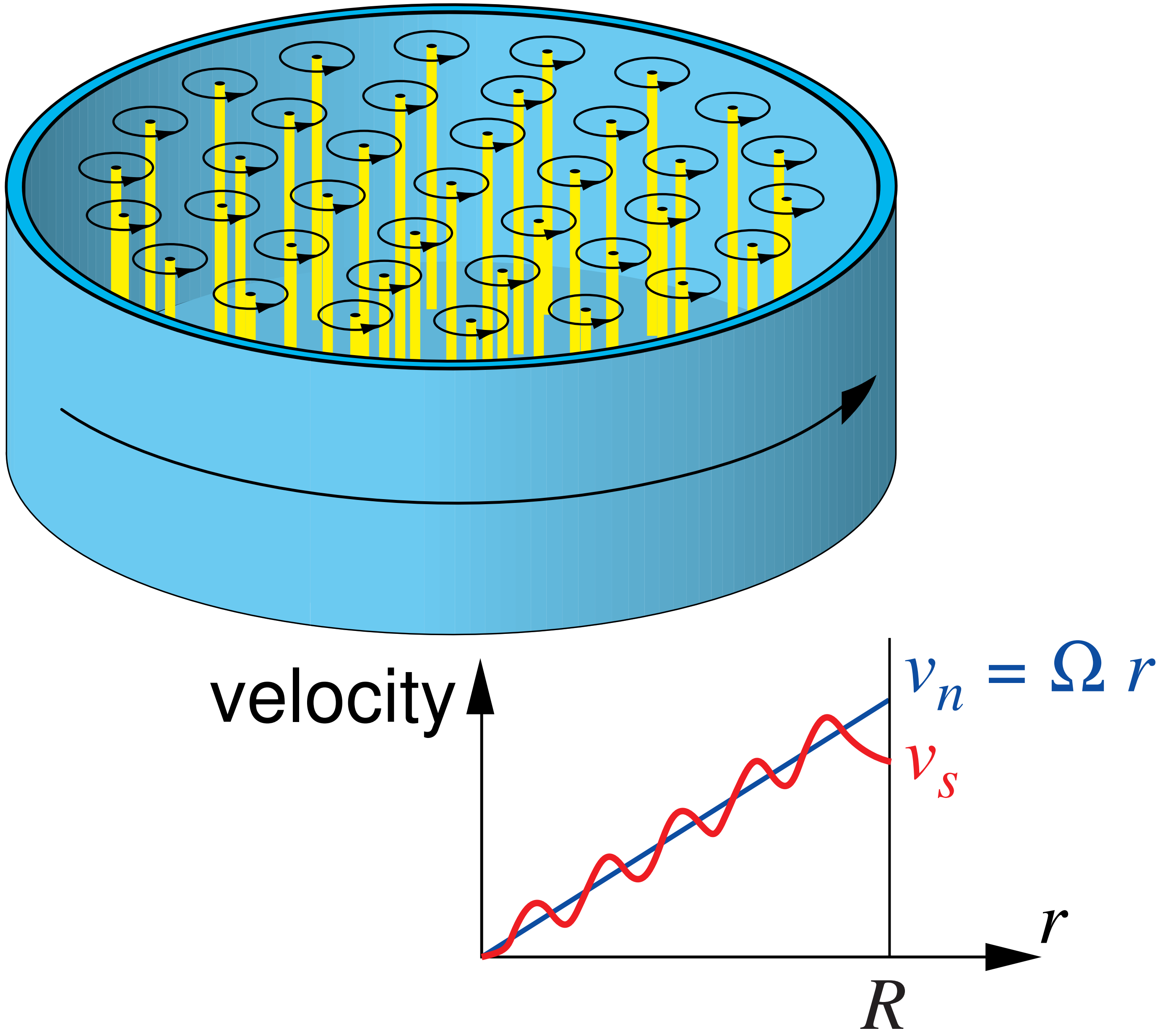




Charged superfluid: Flux line



Vortices in rotating superfluid



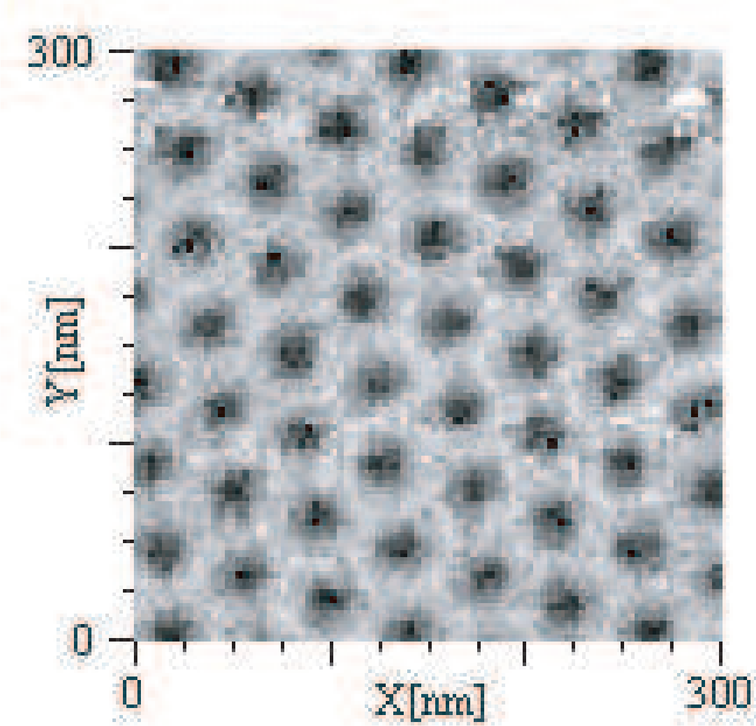
In equilibrium the average superfluid velocity has to be equal to the normal fluid velocity

Line density n

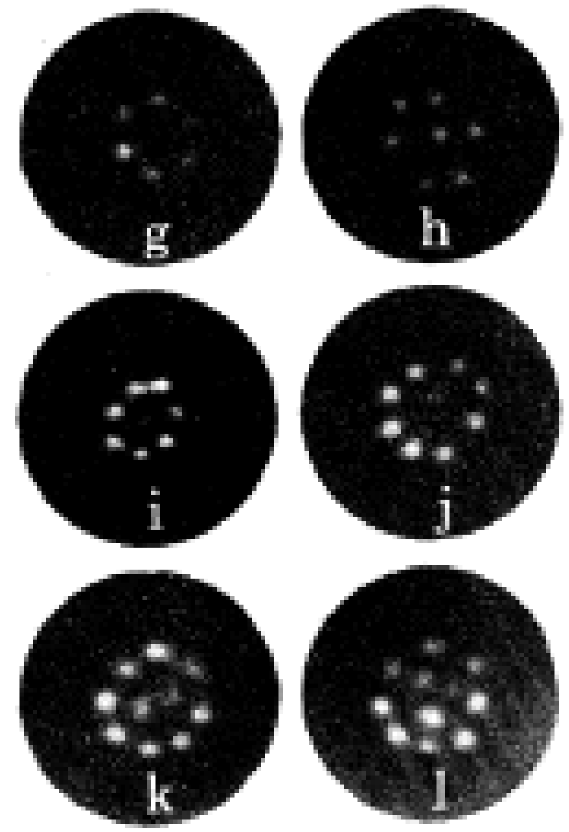
$$\oint d\boldsymbol{l} \cdot \boldsymbol{v}_s = 2\pi R \Omega R = n\kappa\pi R^2$$

$$\Rightarrow n = \frac{2\Omega}{\kappa}.$$

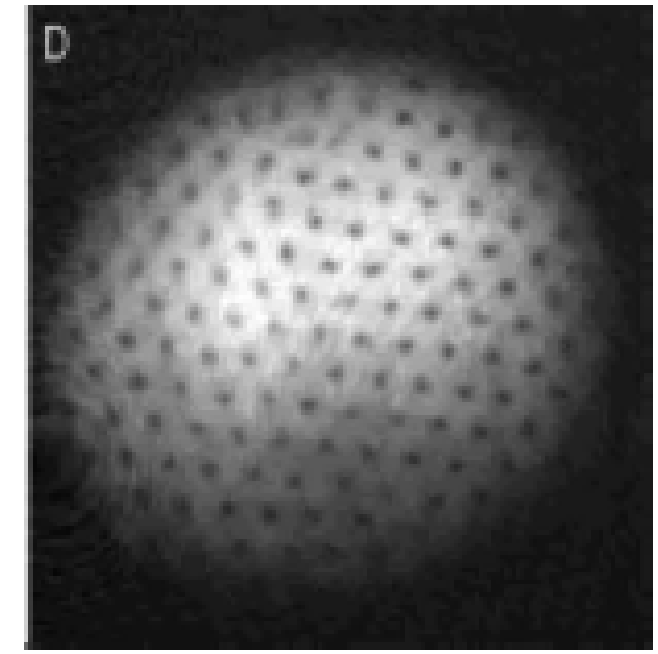
Vortex lattice



Flux lines
in
superconductors
(NbSe$_2$, Ø. Fisher et al)



Vortex lines
in ^{4}He
(Yarmchuk et al 1979)



Vortex lines
in BEC
(Abo-Shaeer et al 2001)

$\Rightarrow$ hexagonal lattice + boundary distortion

Are there other lattice structures?

- underlying crystal lattice
  $\rightarrow$ different vortex lattice structures, or no periodicity
- more complicated order parameter

Multi-component order parameter

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \end{pmatrix}$$

- mixture BEC: two species of atoms
- spinor BEC: atoms identical but in different hyperfine states
- "exotic" superconductors
- ^{3}He

Spin 1 condensate. Use basis $|F=1, m_F\rangle$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} = \sqrt{n}e^{i\phi} \begin{pmatrix} \zeta_1 \\ \zeta_0 \\ \zeta_{-1} \end{pmatrix} \qquad \zeta_\alpha^* \zeta_\alpha = 1$$

$$\langle \boldsymbol{S} \rangle = \zeta_\alpha^* \boldsymbol{S}_{\alpha\beta} \zeta_\beta$$

Two alternatives

1) Ferromagnetic state, $|\langle \boldsymbol{S} \rangle| = 1$. Characterized by direction of $\langle \boldsymbol{S} \rangle$

2) Polar state, $\langle \boldsymbol{S} \rangle = 0$. Convenient representation in basis

$$\Psi = \sqrt{n}e^{i\phi} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \qquad \begin{matrix} \frac{1}{\sqrt{2}}(-|1,1\rangle + |1,-1\rangle) \\ \frac{i}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \\ |1,0\rangle \end{matrix}$$

$$\boldsymbol{v}_s = \frac{\hbar}{M} \boldsymbol{\nabla}\phi$$

The variation of $\boldsymbol{d}$ and $\langle S \rangle$ give rise to spin currents

Symmetries of periodic vortex lattices
in absence of crystal lattice effect

Bravais lattice	space group	material	vortex name
hexagonal	$P6/mm'm'$	^{4}He-II	vortex line
		s.c. metals	flux line
		^{3}He-B	A-phase-core v.
		dilute gas	vortex line
square	$P4/nb'm'$	^{3}He-A	locked vortex 1 (LV1)
prim. rectang.	$Pb'a'n$	^{3}He-A	vortex sheet (VS)
cent. rectang.	$Cm'm'2$	^{3}He-B	double-core v.
	$C2'$	^{3}He-A	cont. unlocked v. (CUV)
		^{3}He-A	locked vortex 2
	Cm'	^{3}He-A	singular vortex (SV)

Basis for symmetry classification

coordinates $\boldsymbol{r}_i$, momenta $\boldsymbol{p}_i=m_i\boldsymbol{v}_i$

$$H = \sum_i \frac{p_i^2}{2m_i} + V + \frac{1}{2} I\Omega^2 + U$$

interactions

confining potential and its kinetic energy (angular velocity Ω)

require $L_{\mathrm{tot}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i + I\boldsymbol{\Omega} = \text{constant}$

define $\boldsymbol{v}_n(\boldsymbol{r}) = \boldsymbol{\Omega} \times \boldsymbol{r}$

⇒

$$H = \underbrace{\sum_i \frac{1}{2m_i}[\boldsymbol{p}_i - m_i \boldsymbol{v}_n(\boldsymbol{r}_i)]^2 + V}_{\to \text{ periodic solutions}} \underbrace{- \frac{1}{2} \sum_i m_i v_n(\boldsymbol{r}_i)^2 + U}_{\text{neglect}}$$

charged superfluid $\boldsymbol{p}_i \to \boldsymbol{p}_i - e_i \boldsymbol{A}(\boldsymbol{r}_i)$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2} \boldsymbol{B} \times \boldsymbol{r}$$

$$H = \sum_i \frac{1}{2m_i}[\boldsymbol{p}_i - m_i \boldsymbol{v}_n(\boldsymbol{r}_i) - e_i \boldsymbol{A}(\boldsymbol{r}_i)]^2 + V - \frac{1}{2} \sum_i m_i v_n(\boldsymbol{r}_i)^2 + U$$

(Equivalence of magnetic field and rotation ⇒ London moment)

Space groups for vortex lattices

There are 230 space groups for 3D crystals

Basic symmetry operations for vortex lattices:
1) translations
2) rotations around Ω (1,2,3,4,6)
3) rotation by π around axis $\perp \Omega$ combined by time inversion (2')
4) reflection in plane $\perp \Omega$ (m)
5) reflection in plane containing Ω combined by time inversion (m')

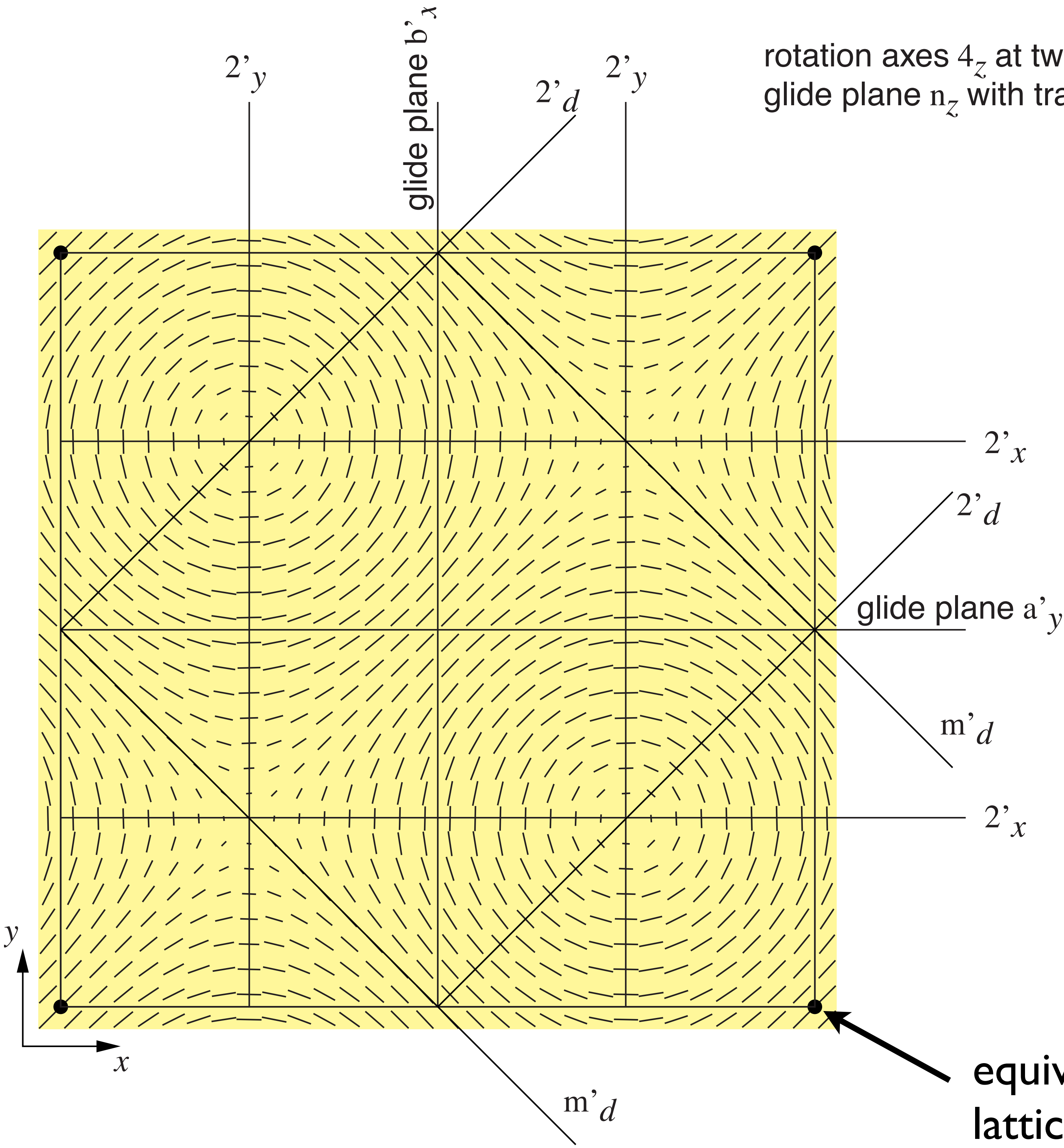
vortex lattice: continuous translation symmetry along the rotation axis (?)
- space groups can still be used, but cubic groups are not possible

Why not 17 plane groups?
- they do not contain operation 4 above
- do not allow breaking of the continuous translation symmetry along rotation axis

Why not 1651 magnetic space groups?
- too complicated since time inversion appears trivially

(Karimäki & Thuneberg, PRB 1999)

An example: P4/nb'm'



rotation axes 4_z at two circular points
glide plane n_z with translation along diagonal

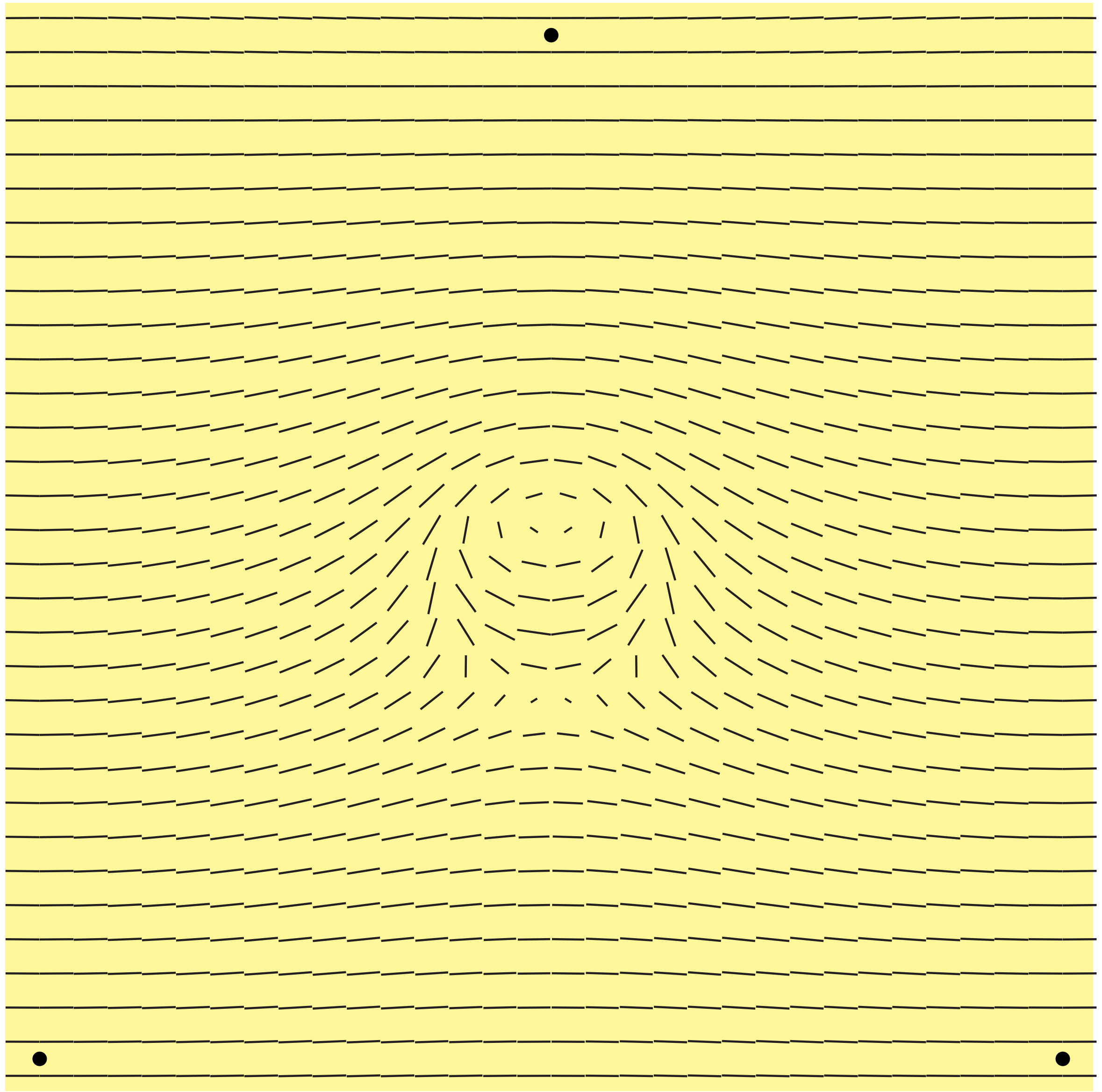
Continuous locked vortex in ^{3}He-A (Fujita, Nakahara, Ohmi & Tsuneto 1978)

The dashes show projection of $\boldsymbol{l}$ vector on the plane (arrow heads are removed for clarity)

$\boldsymbol{l}$ points up at the circular points and down at the hyperbolic points

equivalent points of the Bravais lattice

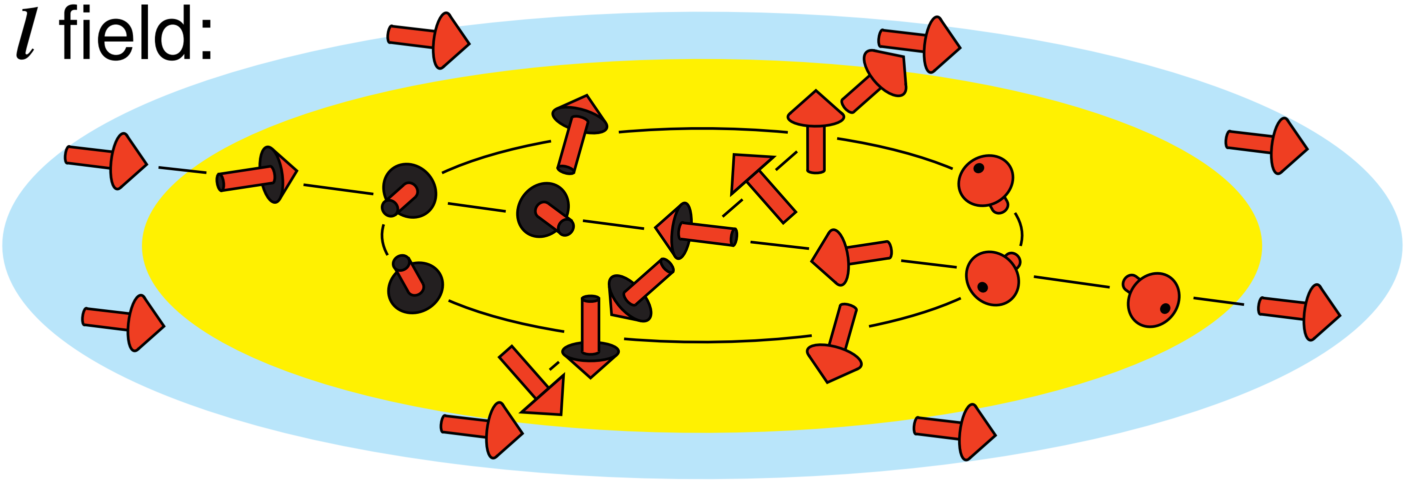
Second example: C2'



Centered rectangular lattice

Continuous unlocked vortex in ^{3}He-A
(Volovik & Seppälä 1983)

$\hat{l}$ field:



Topological classification

In contrast to symmetry classification, topological classification depends on the type of the order parameter

Simple superfluids: circulation around a vortex

$$\kappa = \oint d\boldsymbol{l} \cdot \boldsymbol{v}_s = \frac{\hbar}{M} \oint d\boldsymbol{l} \cdot \boldsymbol{\nabla}\varphi = n\frac{h}{M}.$$

Many-component superfluids: additional topological invariants

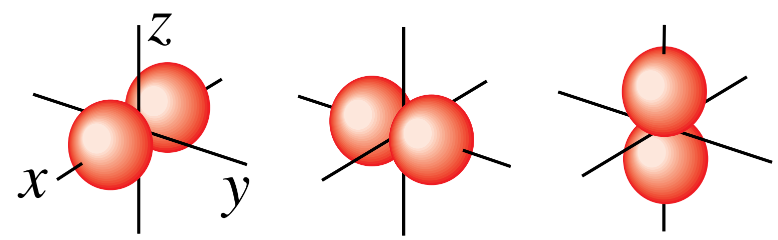
$$\nu_l \quad = \quad \frac{1}{4\pi} \int dx\, dy\ \hat{\boldsymbol{l}} \cdot \frac{\partial \hat{\boldsymbol{l}}}{\partial x} \times \frac{\partial \hat{\boldsymbol{l}}}{\partial y}$$

counts how many times $\boldsymbol{l}$ sweeps the unit sphere

Order parameter of ^{3}He

^{3}He is spin 1/2 fermion, fermions form pairs in state $S=1, L=1$.
Such a Cooper pair state has macroscopic occupation, similar as in Bose condensation

Orbital wave functions (L = 1)



Spin wave functions (S = 1)

$S_x = 0$: (-↑↑+↓↓)

$S_y = 0$: i(↑↑+↓↓)

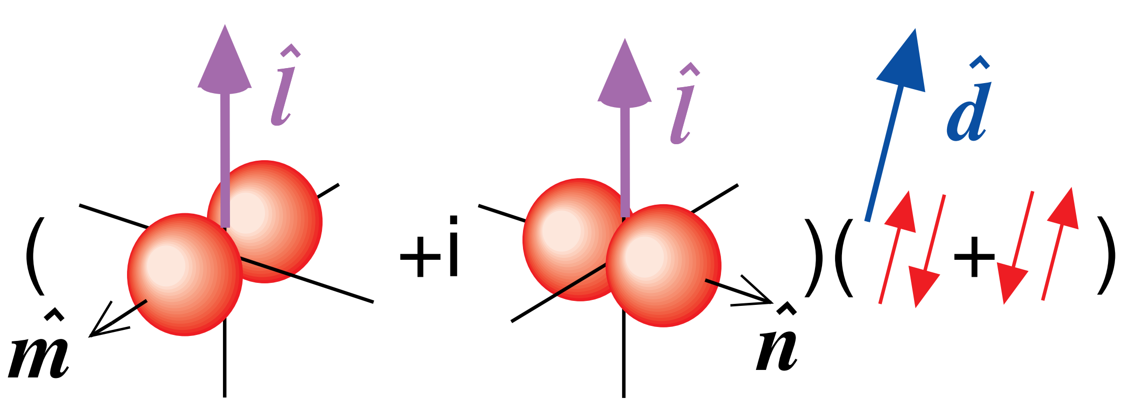
$S_z = 0$: (↑↓+↓↑)

$$\begin{pmatrix} \Psi_{xx} & \Psi_{xy} & \Psi_{xz} \\ \Psi_{yx} & \Psi_{yy} & \Psi_{yz} \\ \Psi_{zx} & \Psi_{zy} & \Psi_{zz} \end{pmatrix}$$

order parameter

$\Psi_{ij}(\boldsymbol{r})$ is the wave function for the center of mass of a Cooper pair.

In the A phase energetics limits the order parameter to the form

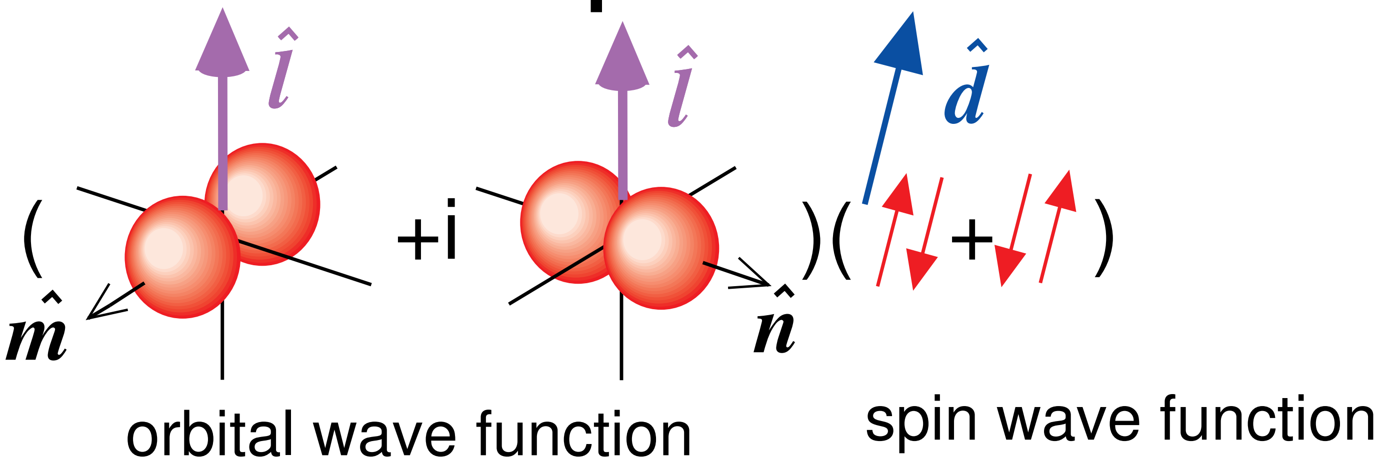


orbital wave function

spin wave function

$$\Psi_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$$

Superfluid velocity in ^{3}He-A



A phase factor $e^{i\phi}$ corresponds to rotation of $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$ around $\hat{\boldsymbol{l}}$:

$$
\begin{aligned}
e^{i\phi}(\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}}) &= (\cos\phi + i\sin\phi)(\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}}) \\
&= (\hat{\boldsymbol{m}}\cos\phi - \hat{\boldsymbol{n}}\sin\phi) + i(\hat{\boldsymbol{m}}\sin\phi + \hat{\boldsymbol{n}}\cos\phi)
\end{aligned}
$$

Superfluid velocity

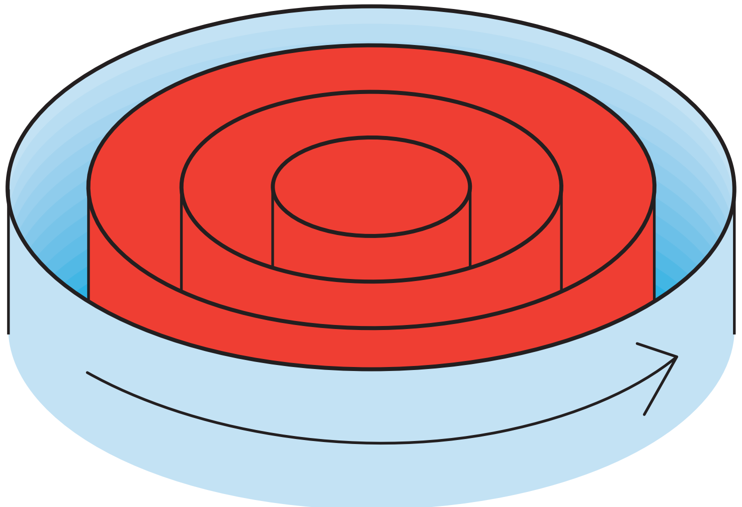
$$
\boldsymbol{v}_{\mathrm{s}} = \frac{\hbar}{M}\boldsymbol{\nabla}\phi = \frac{\hbar}{M}\sum_j \hat{m}_j \boldsymbol{\nabla}\hat{n}_j .
$$

$\Rightarrow$ $\boldsymbol{v}_s$ is coupled to the orientation of the triad $\hat{\boldsymbol{m}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{l}}$

$\Rightarrow$ Mermin-Ho relation $n = 2\nu_l$

$\Rightarrow$ The structures presented above are vortices, although the order parameter vanishes nowhere

Vortex sheet



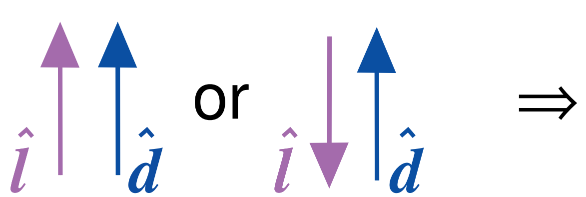
Vortex sheet is a tangential discontinuity in superfluid velocity

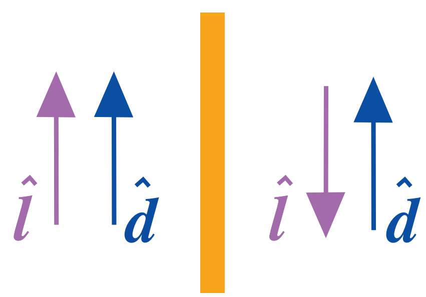
Vortex sheets were discussed before vortex lines:
Onsager 1948, Landau & Lifshitz 1955

Vortex sheets are unstable in simple superfluids

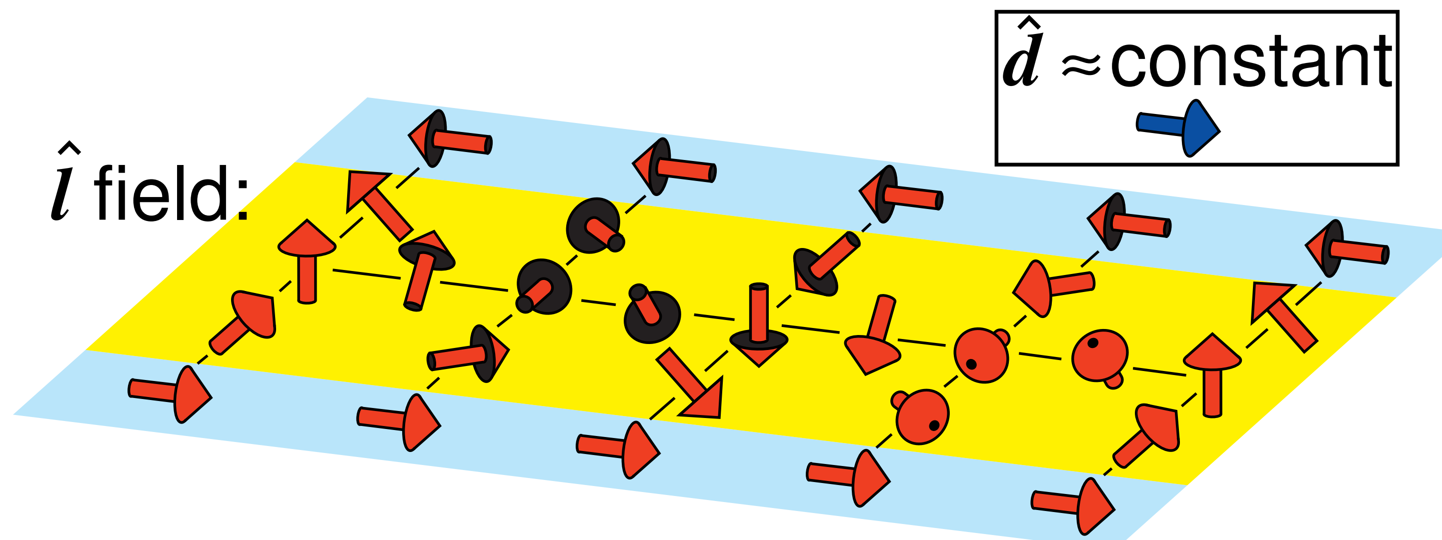
Vortex sheets can be stabilized in multi-component superfluids:
If there are two degenerate but distinct states, there is a domain wall between these.
This domain wall may trap vortices and thus becomes vortex sheet

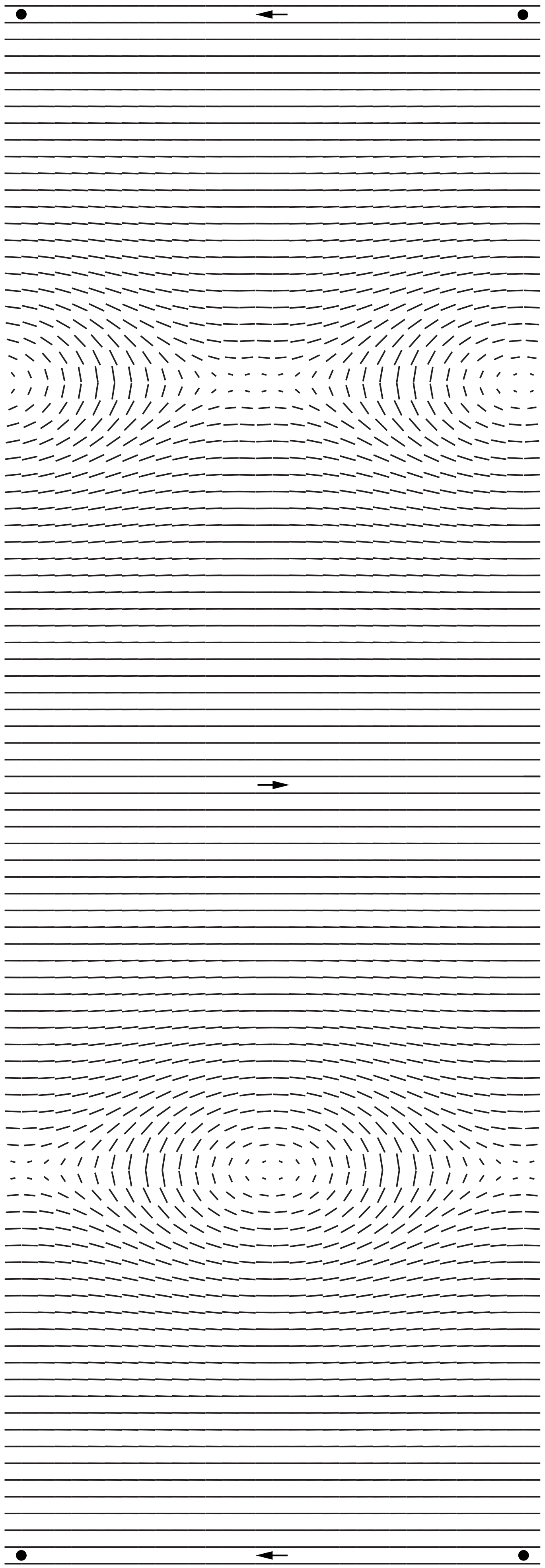
Example in ^{3}He-A





$$f_D = -\frac{1}{2}\lambda_{\rm D}(\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{l}})^2$$





Structure of the sheet

Space group Pb’a’n

Sheet configuration

The equilibrium configuration is determined by the minimum of

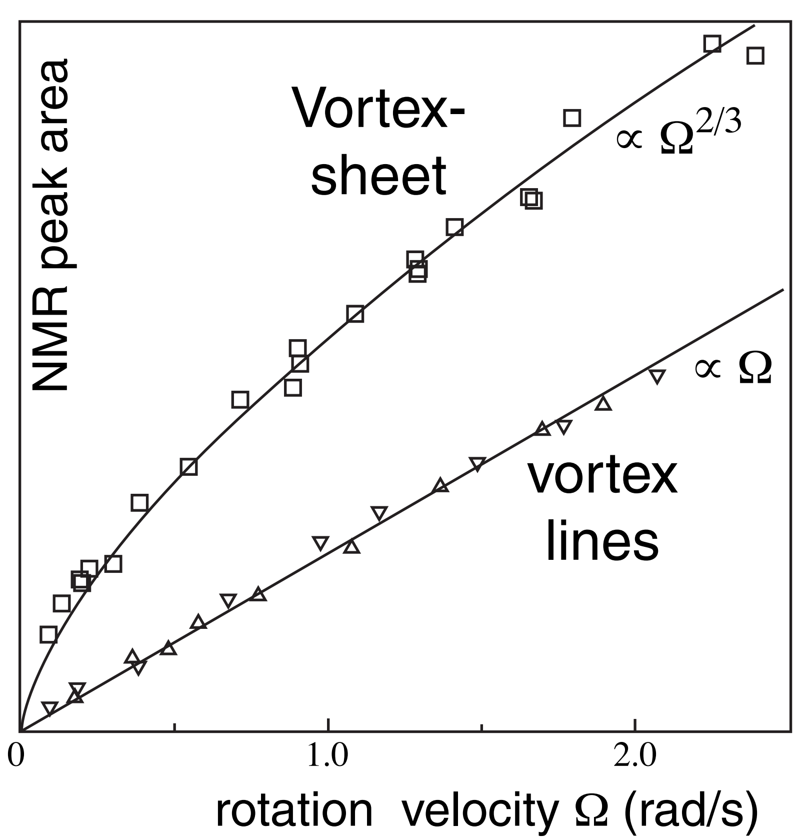
$$F = \int d^3r \, \tfrac{1}{2}\rho_s(\boldsymbol{v}_n - \boldsymbol{v}_s)^2 + \sigma A$$

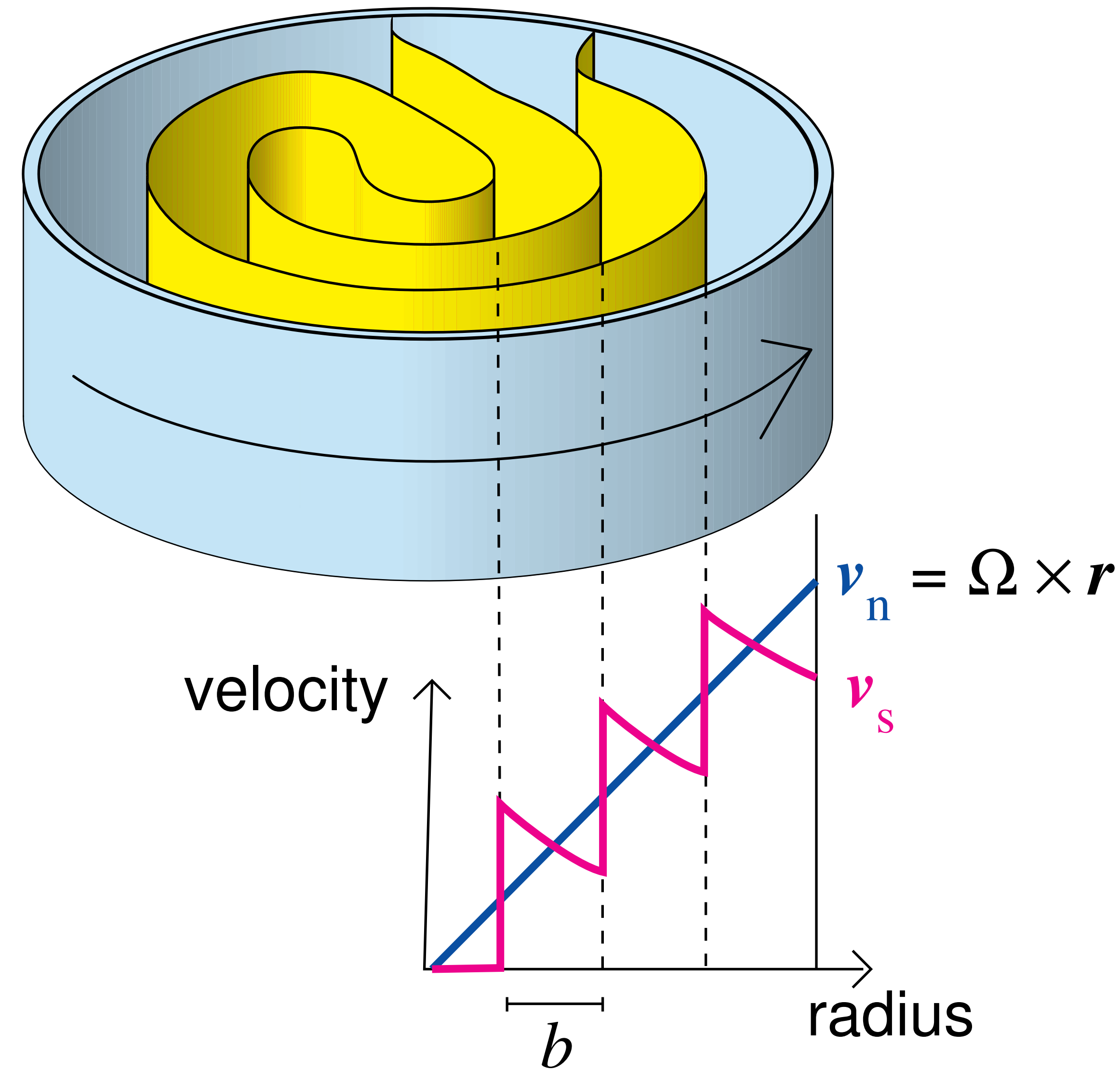
⇒ The equilibrium distance b between sheets

$$b = \left(\frac{3\sigma}{\rho_s \Omega^2}\right)^{1/3}$$

⇒ The total area of the sheet

$$A \propto \frac{1}{b} \propto \Omega^{2/3}$$

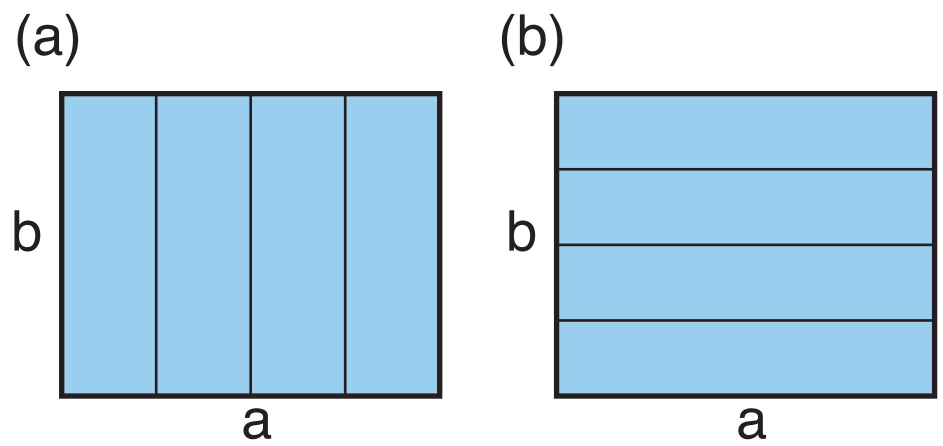




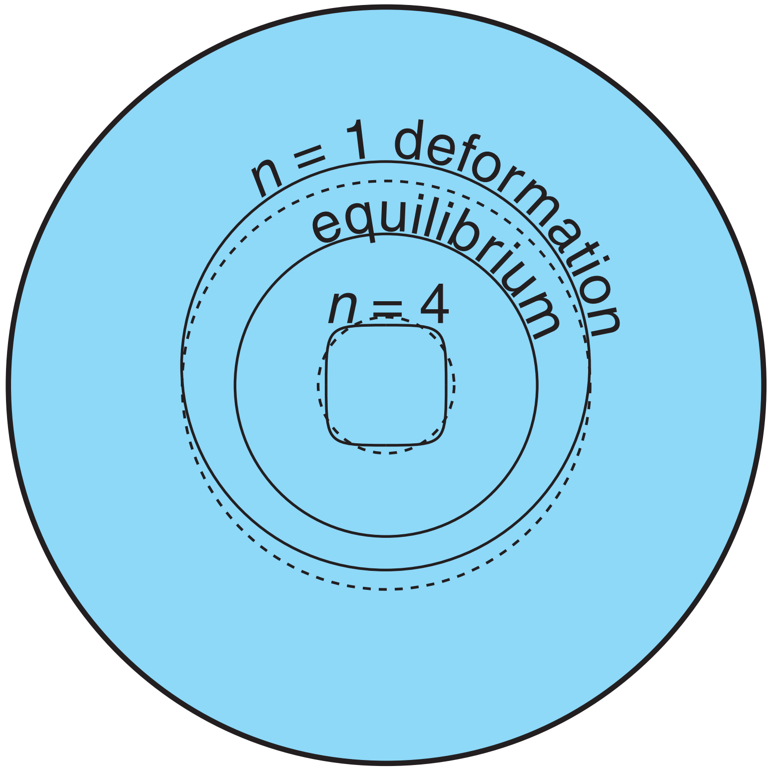
Connection lines with the side wall allow the vortex sheet to grow and shrink when angular velocity changes.

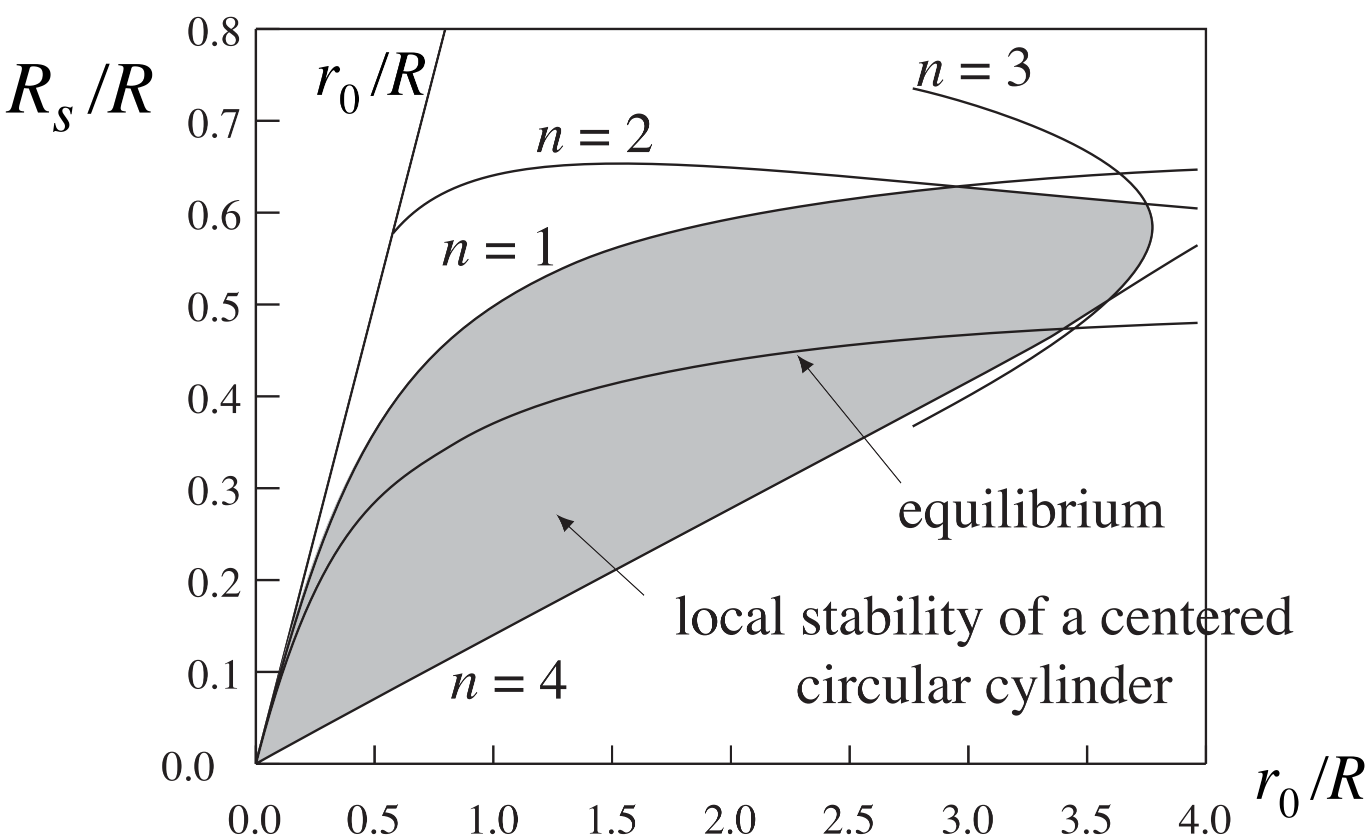
Playing with the sheet

Minimum energy state in a rectangular container



Stability of one concentric vortes sheet in a cylinder



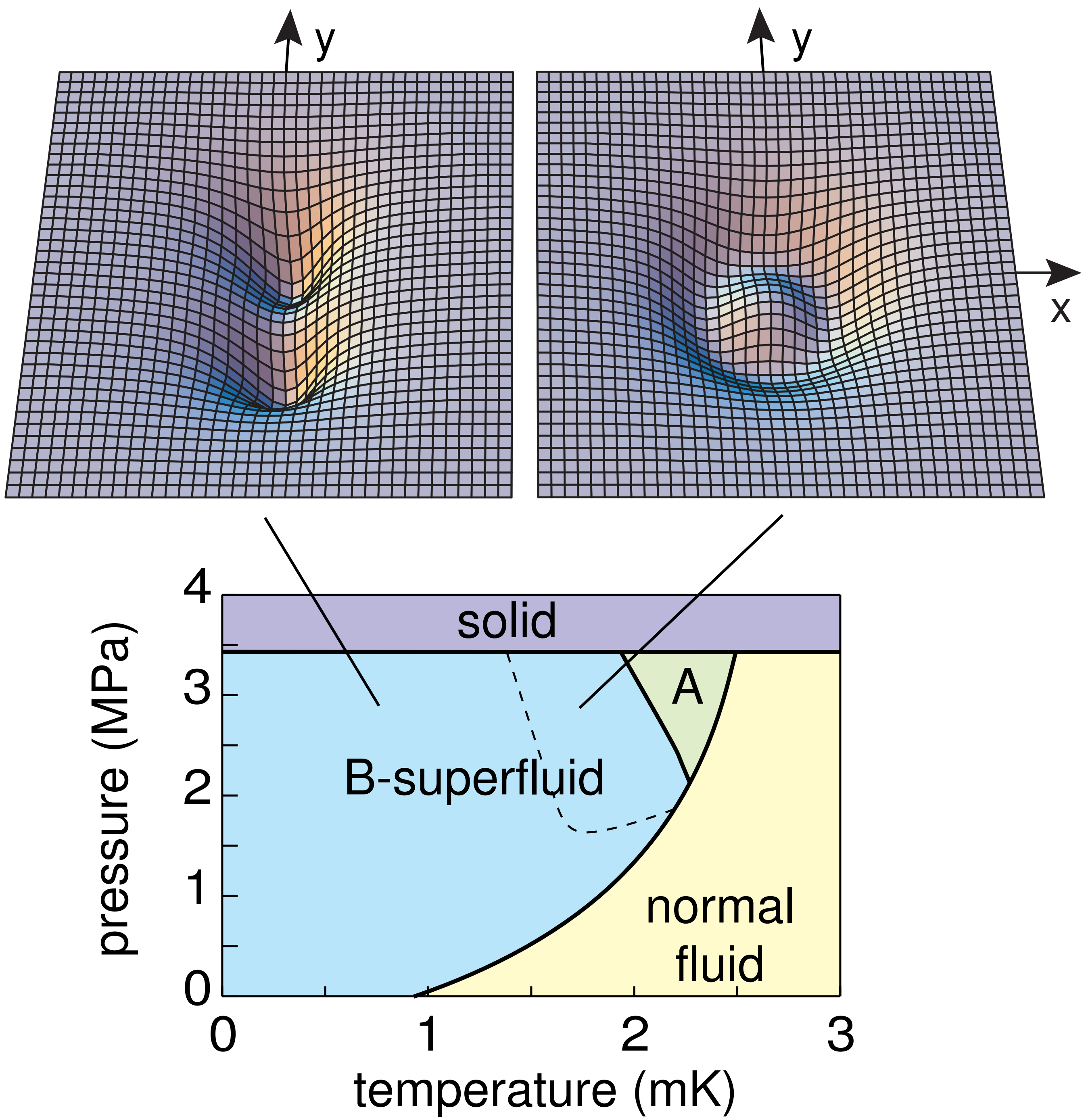


Superfluids with well defined phase

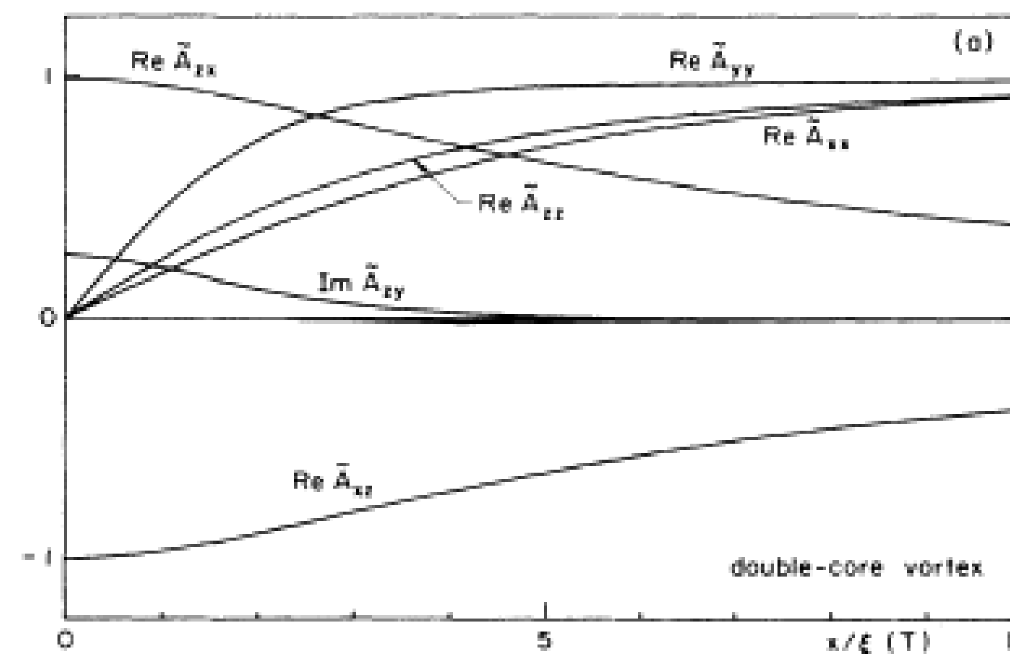
In the case the phase is uniquely defined, the vorticity has line structure, similar to simple superfluids

The structure of the vortex core can be more complicated than in simple superfluids

Example: ^{3}He-B



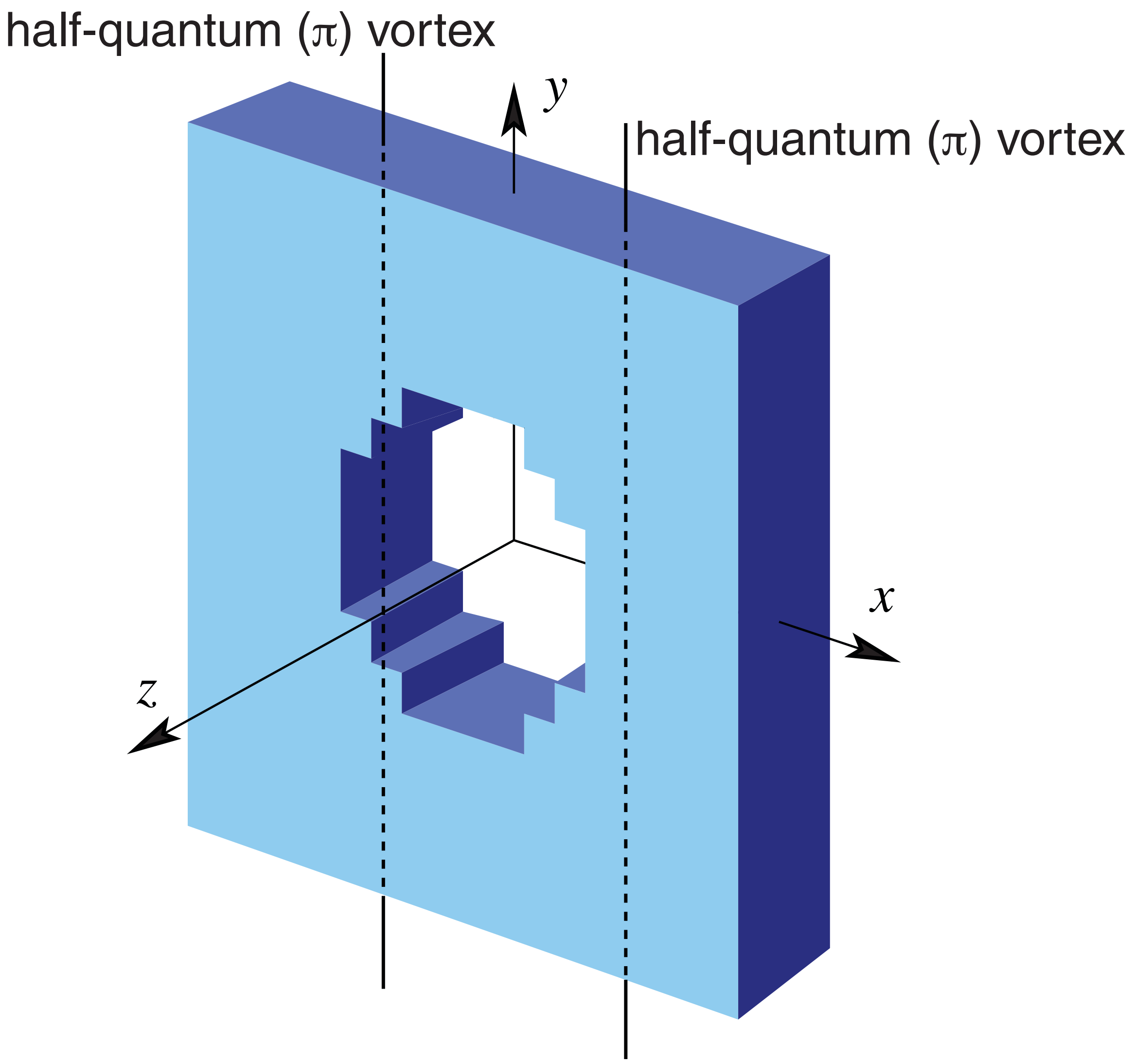
The douple-core structure (left)

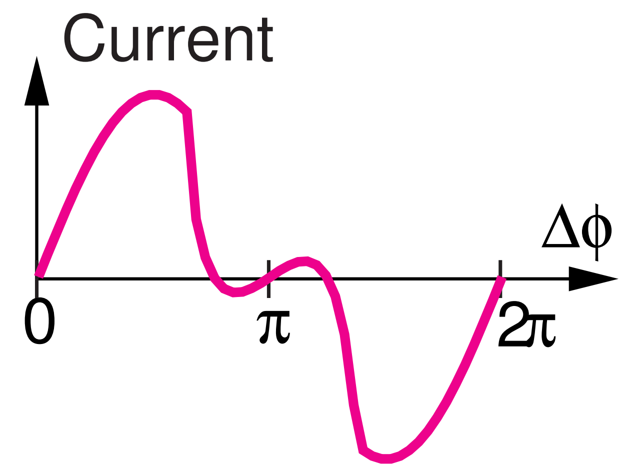


The double-core can be interpreted as two half-quantum vortices

Double core structure in an orifice

The double core structure in an orifice can explain the π state observed in 3He-B Josephson junctions





Spin-current vortices

The vortices discussed above are characterized by mass (or electric) current.

In many-component systems, the variation of spin variables gives rise to spin current.

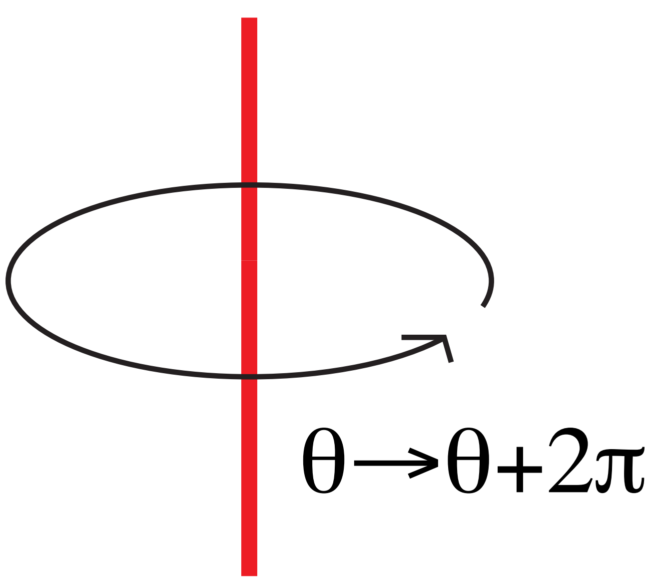
$\Rightarrow$ Spin-current vortex

Example: ^{3}He-B

$$\Psi = \Delta \exp(i\phi) \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

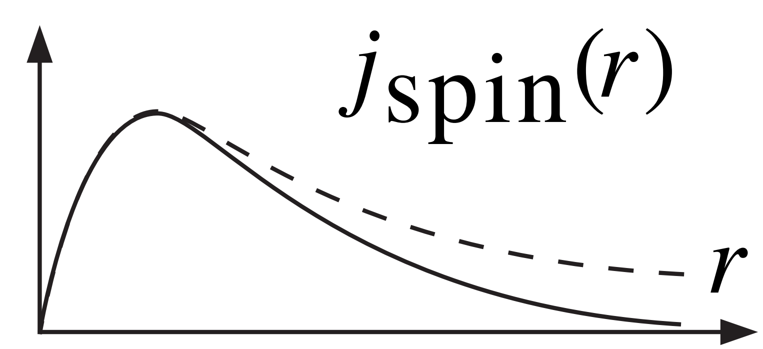
where the rotation matrix $R(\boldsymbol{n}, \theta)$ is paramterized by an angle θ and rotation axis $\boldsymbol{n}$

Symmetry $U(1) \times SO_3$. Homotopy groups $\pi(U(1)) = Z$, $\pi(SO_3) = Z_2$.



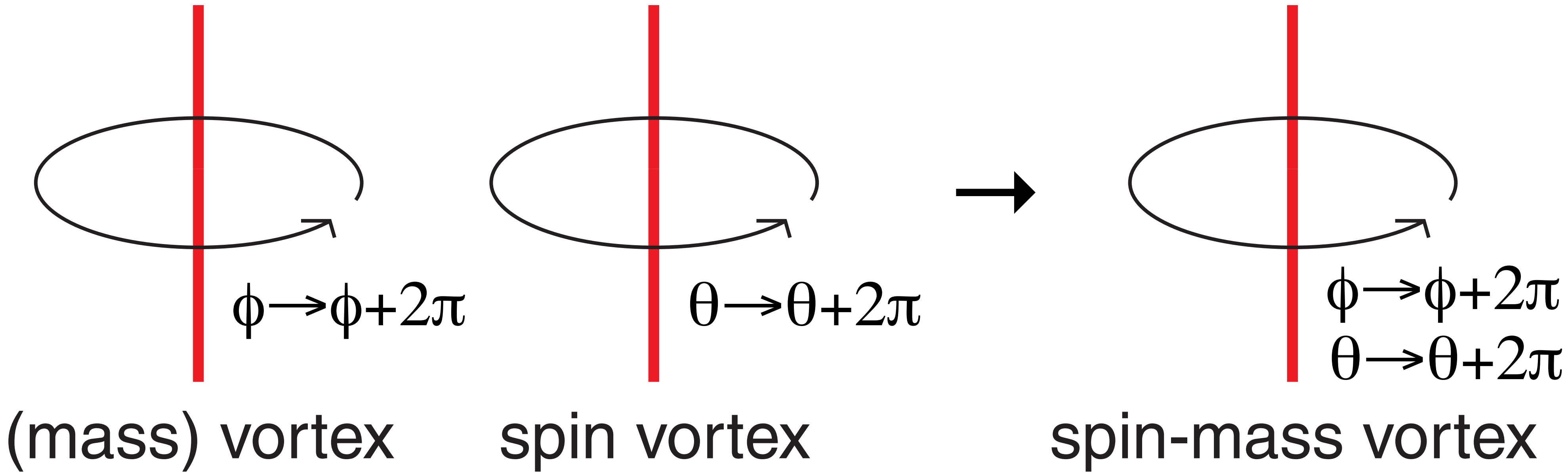
spin vortex

Spin current is not conserved



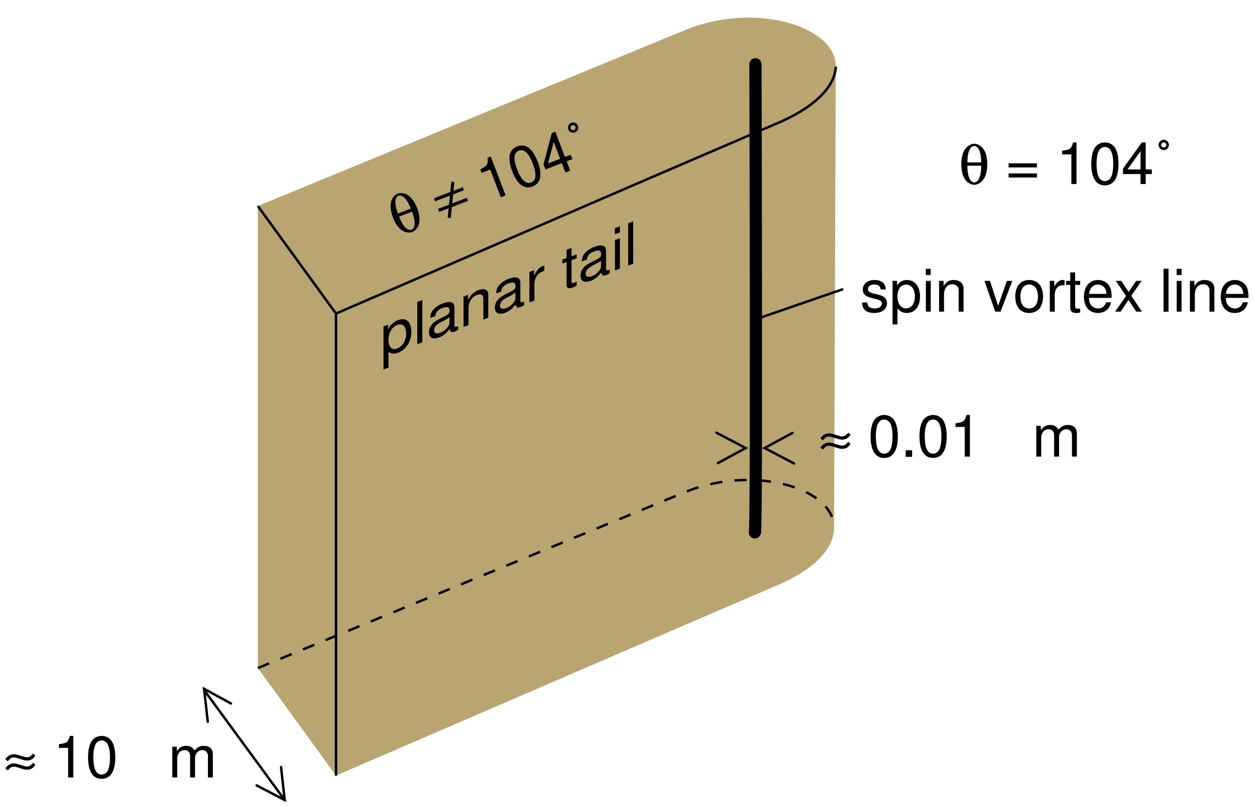
Warning: no expermental evidence of this vortex yet

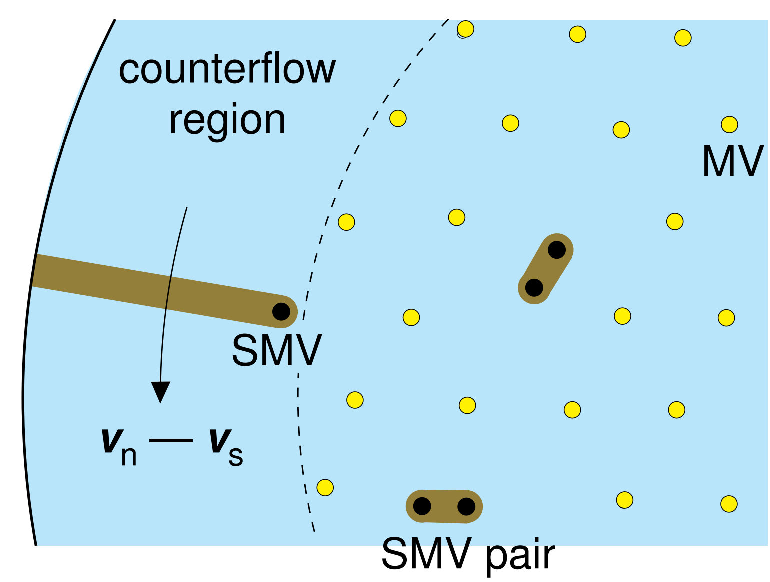
Combined spin and mass vortex, case 1



The spin-mass vortex is stable against dissociation: $F_{\rm sm} < F_{\rm s} + F_{\rm m}$

Effect of non-conservation of spin current:





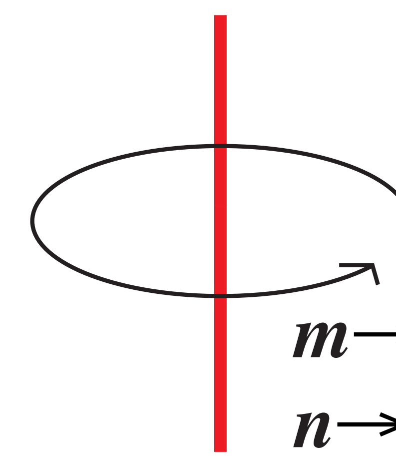
Combined spin and mass vortex, case 2

Assume an order parameter where phase is not uniquely defined

It is possible to have a line object where part of the change of the order parameter around the line comes from spin and part from the phase

Example: half quantum vortex in ^{3}He-A

$$\Psi_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$$



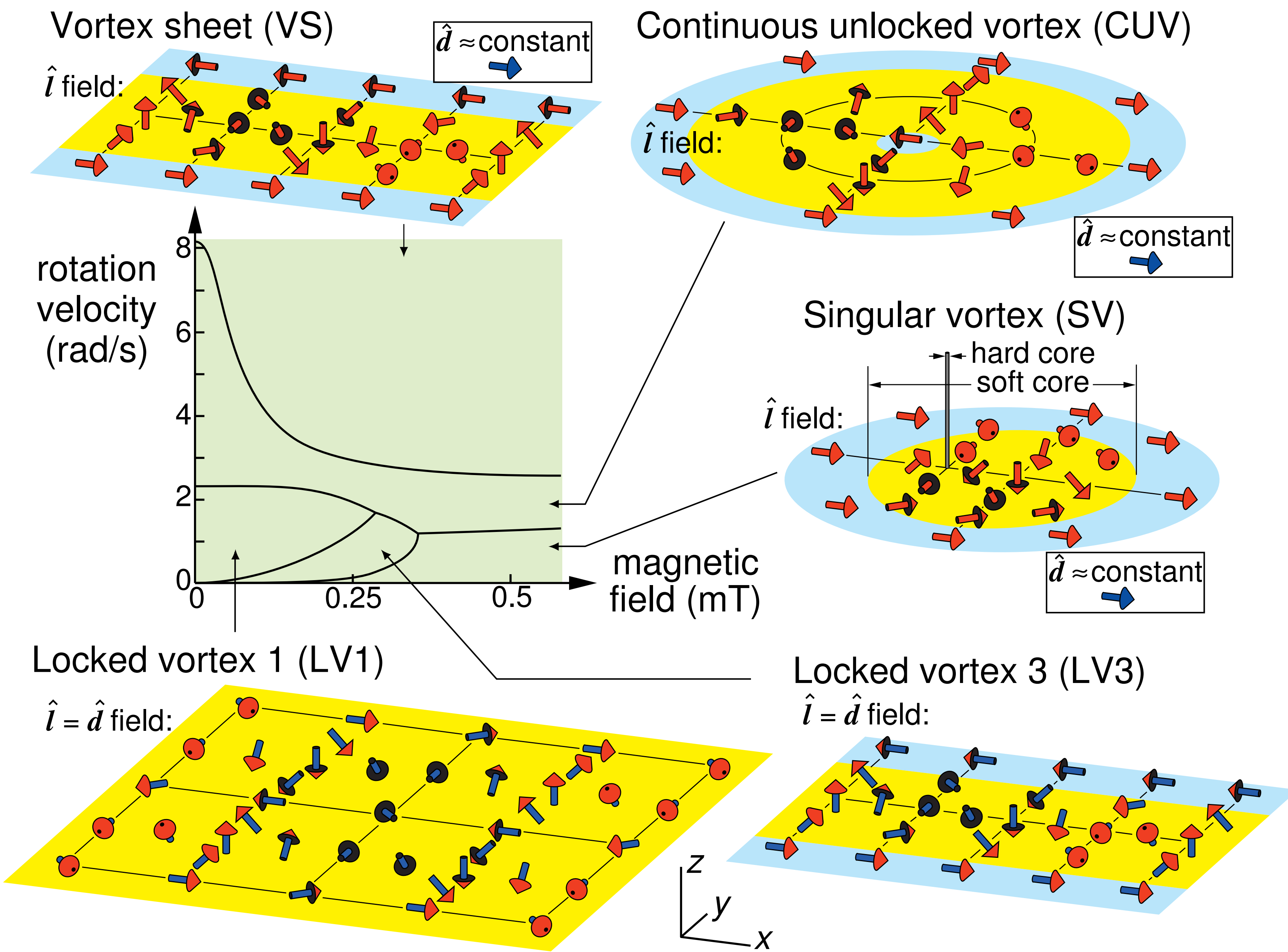
$\boldsymbol{m} \rightarrow -\boldsymbol{m}$
$\boldsymbol{n} \rightarrow -\boldsymbol{n}$
$\boldsymbol{d} \rightarrow -\boldsymbol{d}$

The original order parameter is restored by combined operation:
1) turn ***m*** and ***n*** around ***l*** by angle π (meaning phase change by π)
2) turn ***d*** by angle π

Half quantum vortex not favored in bulk ^{3}He-A, but may be observed in a slab geometry (Mizusaki)

1/3 quantum vortex may appear in some spinor condensates (Machida)

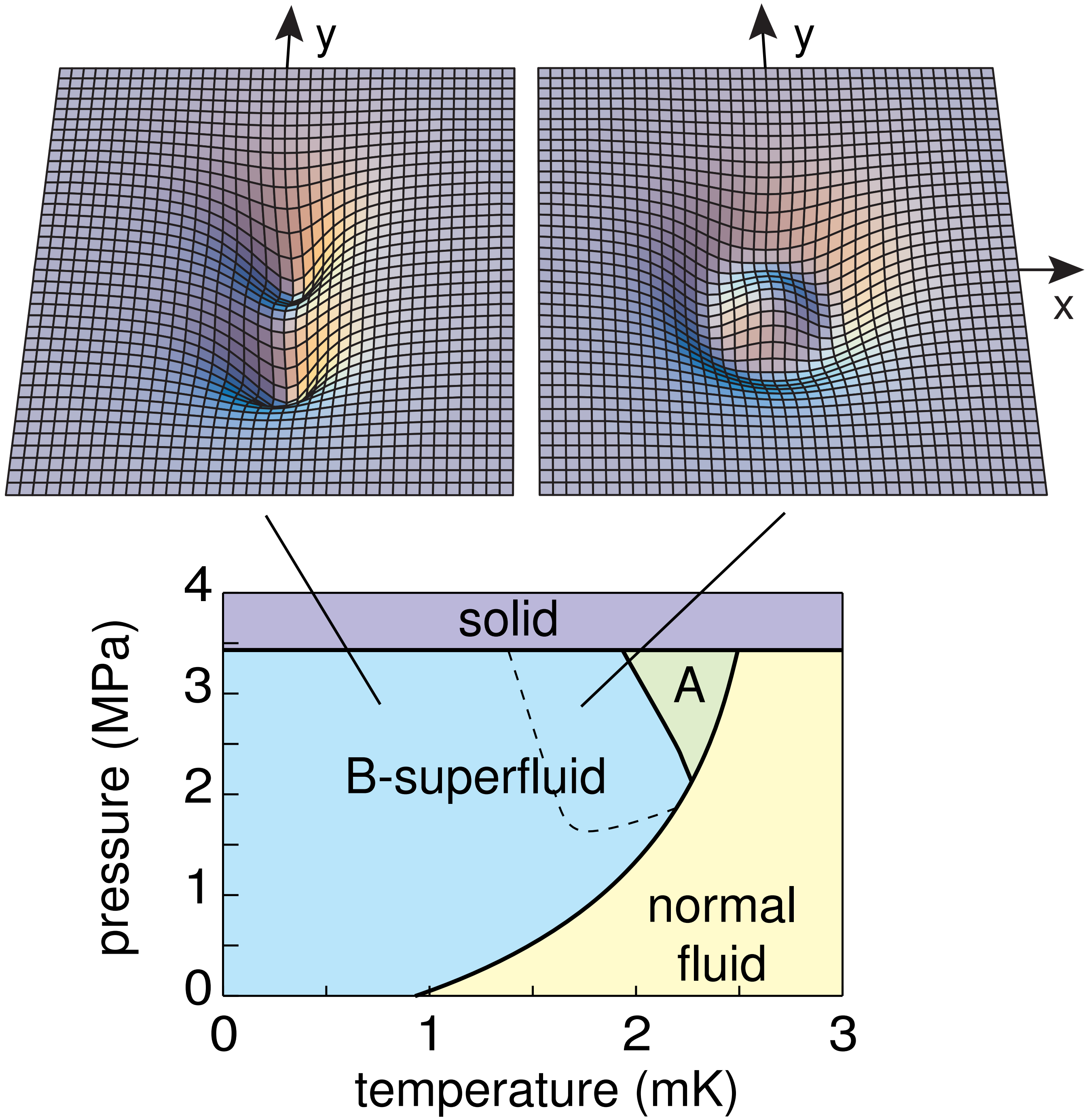
Summary of vortex structures in ^{3}He-A



Vortices in bulk liquid
- are these all at present experimental conditions?
- high rotation speeds (Kita)
- no proper calculation of the singular vortex

Vortices in restricted geometry
- various distorted forms of the bulk structures
- half quantum vortex?
- other structures?

Summary of vortex structures in ^{3}He-B



Equilibrium vortices in bulk liquid: some observations suggest a third structure

- hysteresis at the vortex-core transition (Hall et al)
- gyroscopic experiments see similar but not same transition?

Spin-vortices observed as metastable states

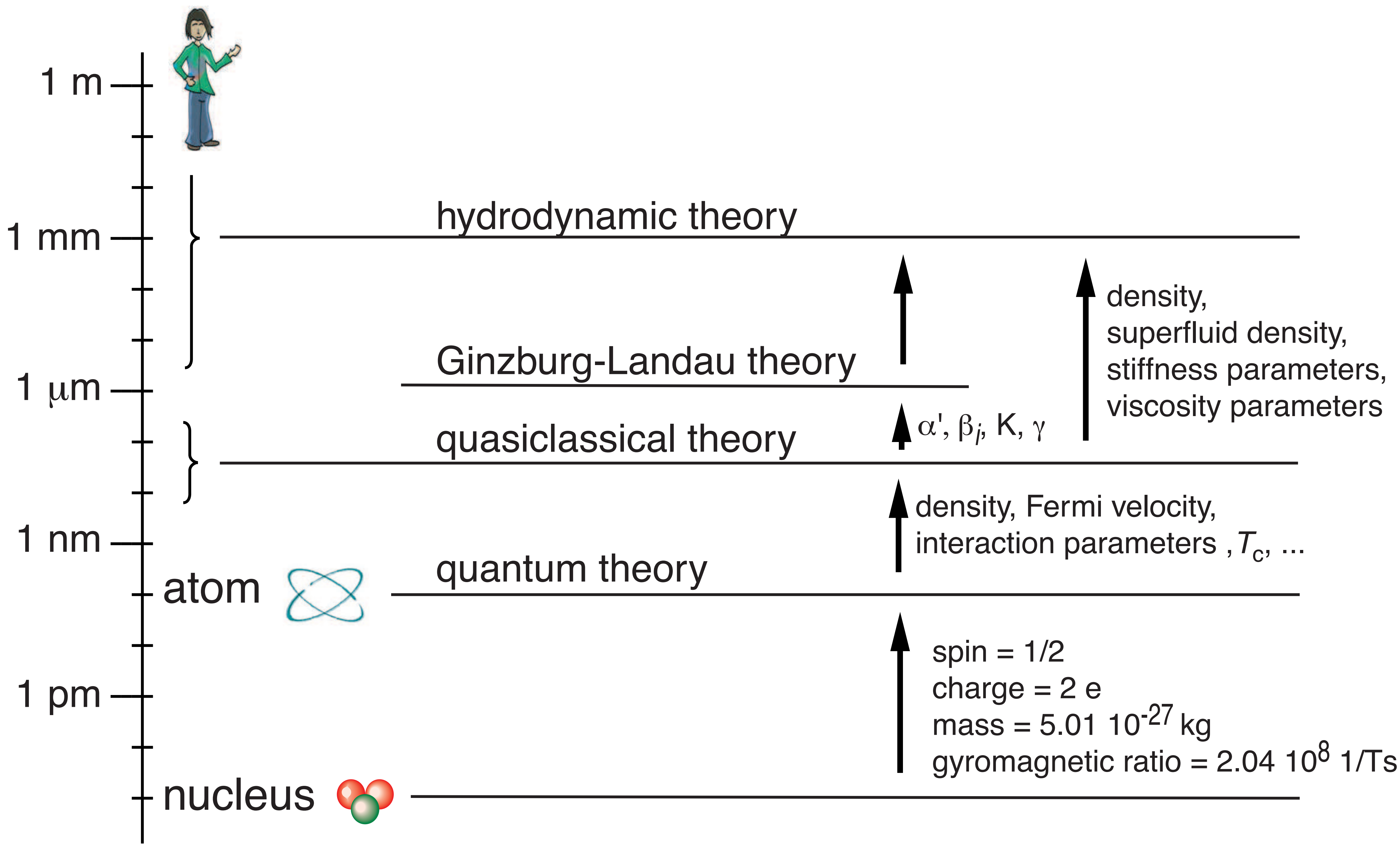
Summary of vortex structures in dilute gases

Summary

Various vortex structures are possible for many-component order-parameter superfluids

These were illustrated by examples in superfluid ^3He

Theory hierarchy of superfluid ^{3}He



Vortex experiments in superfluid ^{3}He